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Multiscale analysis of emulsions and suspensions with surface effects

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Multiscale analysis of emulsions and suspensions with surface effects

by

Grigor Nika

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DEDICATION

Στους γονείς μου, Γιώργο και Έλλη, και στον αδερφό μου Στέλιο.

Abstract

The goal of this thesis is to study multiscale problems dealing with the flow of a viscous fluid containing rigid or fluid particles by variational convergence methods. In particular we focus on fluid emulsions and magnetorheological fluids.

In the first part of the work, Chapters one and two, we consider a periodic emulsion formed by two Newtonian fluids in which one fluid is dispersed under the form of droplets of arbitrary shape, in the presence of surface tension. We assume the droplets have fixed centers of mass and are only allowed to rotate. The non-dilute case was studied by Lipton-Avellaneda [32] and Lipton-Vernescu [33]. Here we are interested in the time-dependent, dilute case when the characteristic size of the droplets a_ϵ , of arbitrary shape, is much smaller than the period length ϵ . The main result of this chapter is the attainment of the limit functional by Γ -convergence. Our approach is supported on two pylons: on the use of Marchenko and Khruslov type of sequences (see [11], [60]) to identify the limit functional, and on an estimate of the stress in terms of the radius of the droplet (see [37]). We obtain a Brinkman type of fluid flow for the critical size $a_\epsilon = \mathcal{O}(\epsilon^3)$ as a replacement of the Stokes flow of the emulsion. Additionally, using Mosco convergence and semigroup theory we extend the convergence to the parabolic case.

For the case when the droplets convect with the flow, it can be shown again using Mosco-convergence that, as the size of the droplets converges to zero faster than the distance between the droplets, the emulsion behaves in the limit like the continuous phase and no “strange” term appears. Moreover, we determine the rate of convergence of the velocity field for the emulsion to that of the velocity for the one fluid problem in both the H^1 and L^2 norms. Additionally, a second order approximation is determined in terms of the bulk and surface polarization tensors for the cases of uniform and non-uniform surface tension. Furthermore, for the case of spherical droplets we recover Taylor’s formula for the emulsion viscosity [57].

The second part of this work is devoted to the study of magnetorheological (MR) flows. We consider a suspension of rigid magnetizable particles in a viscous fluid with an applied external magnetic field. We assume the fluid to be electrically non-conducting. Thus, we use the quasi-static Maxwell equations coupled with the Stokes equations to capture the magnetorheological effect. We upscale using two scale asymptotic expansions to obtain the effective equations consisting of a coupled nonlinear system in a connected phase domain as well as the new constitutive laws. The proposed model generalizes the model in [46] by coupling the velocity of the fluid and the magnetic field intensity. Using the finite element method we compute the effective coefficients for the MR fluid. We analyze the resulting MR model for Poiseuille and Couette flows and compare with experimental data for validation.

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Introduction

The mathematical modeling of composite material requires that we take into account the effects of different length scales. Generally speaking, in a composite the heterogeneities are small compared to the over all dimension of the material. Hence, it is preferable to replace the composite with a “homogeneous” material in view of computational efficiency, where the properties of the homogeneous material depend on the microstructure.

Chapter one and two are devoted to homogenizing an emulsion formed by two newtonian fluids in which one fluid is dispersed under the form of droplets of arbitrary shape, in the presence of surface tension. In Chapter one we consider cases of droplets with fixed centers of mass. We assume that $\Omega \subset \mathbb{R}^3$ is a bounded open set with smooth enough boundary and $Y = (-\frac{1}{2}, \frac{1}{2})^3$ the unit cube in \mathbb{R}^3 . For every $\epsilon > 0$, N^ϵ is the set of all points $\ell \in \mathbb{Z}^3$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. T is the closure of an open connected set with smooth enough boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we considered the set $T_\ell^\epsilon \subset \subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon\ell + a_\epsilon T$, where $a_\epsilon \ll \epsilon$. The set T_ℓ^ϵ represents one of the droplets suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface. Moreover, we defined the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}},$$

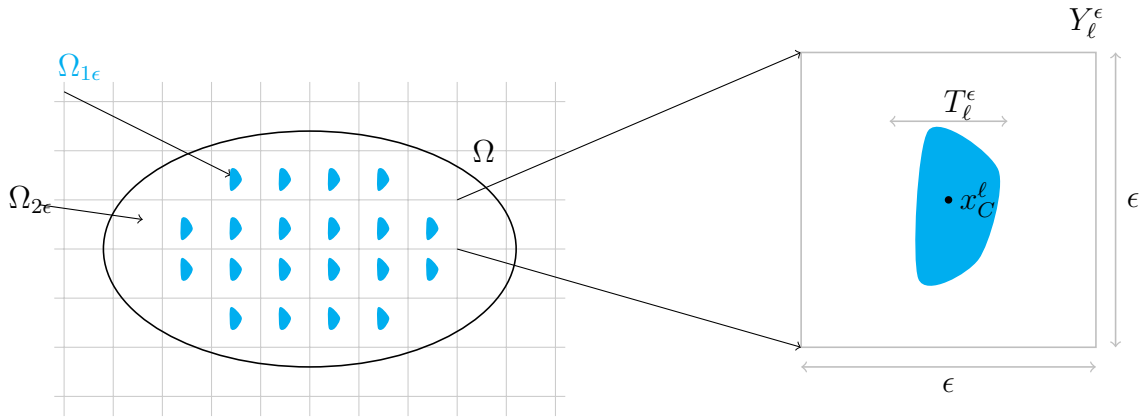


Figure 1: Schematic of the emulsion and the unit cell

where $\Omega_{1\epsilon}$ is the domain occupied by the droplets of viscosity μ_1 , and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid, of viscosity μ_2 . The unit normal on the boundary of $\Omega_{2\epsilon}$ that points outside the domain will be denoted by \mathbf{n} .

The mathematical description of the above problem is to find $(\mathbf{v}^\epsilon, p^\epsilon) \in H_0^1(\Omega)^3 \times L^2(\Omega)/\mathbb{R}$ solution of

$$\begin{aligned} -\operatorname{div} \sigma^\epsilon &= \mathbf{f} \text{ in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \\ \sigma^\epsilon &= -p^\epsilon I + 2\mu^\epsilon e(\mathbf{v}^\epsilon), \\ \operatorname{div} \mathbf{v}^\epsilon &= 0 \text{ in } \Omega, \end{aligned} \tag{0.0.1}$$

with boundary conditions on the surface of each droplet T_ℓ^ϵ , $\ell \in N^\epsilon$:

$$\begin{aligned} \llbracket \mathbf{v}^\epsilon \rrbracket &= 0 \quad \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \quad \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= 0 \quad \text{on } \partial\Omega, \\ \int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \sigma^\epsilon \mathbf{n} \rrbracket ds &= 0, \end{aligned} \tag{0.0.2}$$

where $\llbracket \cdot \rrbracket$ denotes the jump across S_ℓ^ϵ , $\boldsymbol{\omega}^{\ell, \epsilon}$ is an unknown, constant vector in \mathbb{R}^3 , \mathbf{x}_C^ℓ the position vector of the center of mass of the droplet T_ℓ^ϵ , and the viscosity μ^ϵ defined as

$$\mu^\epsilon(\mathbf{x}) = \begin{cases} \mu_1 & \text{if } x \in \Omega_{1\epsilon}, \\ \mu_2 & \text{if } x \in \Omega_{2\epsilon}. \end{cases}$$

Solutions to the previous system can be characterized as a unique minimizers of:

$$\begin{cases} \text{Find } \mathbf{v}^\epsilon \in H_0^1(\Omega)^3 \text{ such that,} \\ J^\epsilon(\mathbf{v}^\epsilon) = \min_{\mathbf{u} \in H_0^1(\Omega)^3} J^\epsilon(\mathbf{u}), \end{cases}$$

where

$$J^\epsilon(\mathbf{u}) = \int_{\Omega} \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}),$$

I_S represents the indicator function of the set S , defined by

$$I_S(s) = \begin{cases} 0 & \text{if } s \in S \\ +\infty & \text{if } s \notin S \end{cases}$$

and V^ϵ is the closed subspace of $H_0^1(\Omega)^3$ given by:

$$V^\epsilon = \left\{ \mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \text{ on } S_\ell^\epsilon, \quad \boldsymbol{\omega}^{\ell, \epsilon} \in \mathbb{R}^3 \right\}.$$

Roughly speaking there are three possible outcomes that can occur when ϵ tends to 0 (see [13]): either the droplets, in spite of their number, are too small and \mathbf{v}^ϵ converges to the solution of an unperturbed Stokes flow type of problem in Ω , or the droplets are too big and \mathbf{v}^ϵ converges to zero. In addition to these two cases, there is a third case, on which we focus on i.e. the case when the droplets are of critical size depending on their number and distribution, and where the limit of \mathbf{v}^ϵ is the solution to a Stokes type of problem with an additional “strange” term in Ω .

The main result of this work was to obtain the limit functional of J^ϵ by Γ -convergence. Our approach is supported on two pylons: on the use of Marchenko and Khruslov type of sequences (see [11], [60]) to identify the limit functional of J^ϵ , and on an estimate of stress in terms of the radius of the droplet (see [37]). Using the Γ -convergence of J^ϵ we get a Brinkman type of fluid flow for the homogeneous fluid replacing the coupled Stokes flow for the emulsion and thus extending the results previously obtained in [11] and [1]. We summarize the main result in the following theorem:

THEOREM 0.0.1. *Let $a_\epsilon = \mathcal{O}(\epsilon^3)$. Then the sequence of functionals*

$$J^\epsilon: H_0^1(\Omega)^3 \mapsto \mathbb{R} \cup \{+\infty\}, \text{ with } J^\epsilon(\mathbf{u}) = \int_{\Omega} \mu^\epsilon e(\mathbf{u}): e(\mathbf{u}) d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_{V^\epsilon}(\mathbf{u})$$

Γ -converges with respect to the weak topology of $H_0^1(\Omega)^3$ to $J: H_0^1(\Omega)^3 \cup \{+\infty\} \mapsto \mathbb{R}$ defined by

$$J(\mathbf{u}) = \int_{\Omega} \mu_2 e(\mathbf{u}): e(\mathbf{u}) d\mathbf{x} + \int_{\Omega} \mathcal{M} \mathbf{u} \cdot \mathbf{u} d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_V(\mathbf{u}),$$

where $V = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}$, and \mathcal{M} is a positive definite, symmetric, second order tensor that depends on the geometry of the droplets and on the two fluid viscosities.

The tensor \mathcal{M} turns out to be a “capacity” defined the following way,

$$\mathcal{M}_{mk} = \lim_{\epsilon \rightarrow 0} \frac{a_\epsilon}{\epsilon^3} \int_{B_\ell^{\frac{\epsilon}{a_\epsilon}}} \mu e(\chi^{m \frac{\epsilon}{a_\epsilon}}): e(\chi^{k \frac{\epsilon}{a_\epsilon}}) d\mathbf{x}$$

where $\chi^{m \frac{\epsilon}{a_\epsilon}}$ is a solution of the unit cell problem.

Brinkman’s law obtained by [11] and [1] for the flow in low porosity porous media can be obtained from here when $\mu_1 \rightarrow +\infty$. For the problem of convected droplets, the limit corresponds to the unperturbed flow, i.e. $\mathcal{M} \equiv 0$.

As an application we considered the case of spherical droplets and in this case the explicit form of the tensor \mathcal{M} is found:

$$\mathcal{M}_{mk} = m\mu_2 \frac{\pi}{8} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \delta_{mk},$$

with $m = \lim_{\epsilon \rightarrow 0} \frac{a_\epsilon}{\epsilon^3}$. Moreover, in the case of suspensions of spherical rigid particles, the tensor reduces to

$$\mathcal{M}_{mk} = m\mu_2 \frac{3\pi}{8} \delta_{mk}.$$

The time dependent case is treated via non-linear semi-group theory and Mosco convergence. To be more precise, the results obtained for the time stationary Stokes equation governing the emulsion can be extended to the time dependent case by using the fact that Γ -convergence of the functionals describing the periodic problem in the weak topology $(H_0^1(\Omega))^3$ is equivalent to Mosco convergence of the functionals describing the periodic problem in $(L^2(\Omega))^3$. One then constructs the following evolution problem,

$$\begin{cases} \frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} + \partial E^\epsilon(\hat{\mathbf{u}}^\epsilon(t)) \ni \hat{\mathbf{f}}(t), & 0 < t < T \\ \hat{\mathbf{u}}^\epsilon(0) = \mathbf{u}^\epsilon. \end{cases}$$

where $\hat{\mathbf{u}}^\epsilon : [0, T] \mapsto L^2(\Omega)^3$ is an associated mapping to $\mathbf{u}^\epsilon(t, \mathbf{x})$ defined by $[\hat{\mathbf{u}}^\epsilon(t)](\mathbf{x}) := \mathbf{u}^\epsilon(t, \mathbf{x})$, and E^ϵ is the kinetic energy of the fluid. Using the connection between Mosco convergence [40] and the convergence of the solution to the above evolution problem we obtain convergence in the time dependent case.

Chapter two is devoted to determining the speed of convergence of the velocity field for the emulsion to that of the velocity for the one fluid problem in both L^2 and H^1 norms. In homogenization theory it is of paramount importance to be able to determine the accuracy of the approximating problems. A second order approximation is determined in terms of the bulk and surface polarization tensors.

Often problems in which surface rheological effects are important also involve surfactant transport. Surfactants tend to accumulate along an interface. When they do so, they tend to reduce the surface tension and impart surface rheological properties to the interface. In chapter 2 we consider the previous system with the following changes in the boundary conditions:

1. we obtain the stress jump from the principle that the forces on an element of inter-facial area of arbitrary shape and size must be in equilibrium, because the interface is assumed to have zero thickness and thus zero mass (APPENDIX A.4). Thus, $[[\sigma \mathbf{n}]] = \lambda \kappa \mathbf{n} - \nabla_s \lambda$ where λ is the surface tension, $\nabla_s = \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla)$ is the surface gradient operator, and κ is the mean curvature.

2. the droplets convect with the flow and therefore the kinematic boundary condition becomes $\mathbf{v}^\epsilon = \mathbf{V}^{\ell, \epsilon} + \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell)$ on S_ℓ^ϵ , Hence, we obtain,

$$\begin{aligned} -\operatorname{div}(2\mu^\epsilon e(\mathbf{v}^\epsilon) - p^\epsilon I) &= \mathbf{f} + (\lambda_\epsilon \kappa_\epsilon \mathbf{n} - \nabla_s \lambda_\epsilon) \chi_{S^\epsilon} && \text{in } \Omega, \\ \operatorname{div} \mathbf{v}^\epsilon &= 0 && \text{in } \Omega, \\ [[\mathbf{v}^\epsilon]] &= 0 && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{V}^{\ell, \epsilon} + \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \tag{0.0.3}$$

where χ_{S^ϵ} is a characteristic function, λ_ϵ is the surface tension, κ_ϵ the mean curvature, and \mathbf{n} the exterior normal to the droplets. The above system is in equilibrium since the balance of forces and torques are automatically satisfied as has been shown in [42].

As was mentioned above there are two elements that characterize the nature of this chapter: the speed of converge of the velocity for the emulsion to that of the velocity for the one fluid problem, and a second order approximation in terms of bulk and surface polarization tensors. The convergence of velocity is obtained by the use of Marchenko and Khruslov [35] type of sequences which allow us to have the proper variational formulation over the closed subspace of $H_0^1(\Omega)^3$

$$\mathcal{W}^\epsilon = \left\{ \mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \mathbf{V}^{\ell, \epsilon} + \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \text{ on } S_\ell^\epsilon, \quad \boldsymbol{\omega}^{\ell, \epsilon} \in \mathbb{R}^3 \right\}$$

The correction terms, on the other hand, involve a scaling of the surface tension with a_ϵ so that the surface energy remains bounded. We proposed the following scaling:

$$\lambda_\epsilon(\mathbf{x}) = \lambda_\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) = a_\epsilon \lambda(\mathbf{y}) \quad \text{for all } \mathbf{y} \in S_\ell.$$

where $\lambda(\mathbf{y})$ is the surface tension on the unscaled droplet surface. This is in agreement with Tolman's scaling when the surface tension is uniform [59].

The main results of the chapter can be synopsized in the following two theorems:

THEOREM 0.0.2. *There exists a constant $C > 0$, independent of ϵ , such that*

$$\|\mathbf{v}^\epsilon - \mathbf{v}\|_{H_0^1(\Omega)^3} \leq C a_\epsilon^{3/2}.$$

where \mathbf{v} is the solution to the following Stokes system

$$\begin{aligned} -\operatorname{div} (2 \mu_2 e(\mathbf{v}) - p I) &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 \quad \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0} \quad \text{on } \Gamma. \end{aligned}$$

THEOREM 0.0.3. *For any $\mathbf{z} \in \Omega$ at a distance $d > 0$ away from T^ϵ we have*

$$v_i^\epsilon(\mathbf{z}) = v_i(\mathbf{z}) + a_\epsilon^3 (e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \mathbb{P} e_x(\mathbf{v})(\mathbf{x}_c) - e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \mathbb{S}) + \mathcal{O} \left(a_\epsilon^{3+\frac{1}{2}} \right),$$

where \mathbb{P} and \mathbb{S} are the bulk and surface polarization tensors defined by:

$$\mathbb{P}_{ijkl} = 2(\mu_2 - \mu_1) \int_T e_{yij}(\boldsymbol{\phi}^{kl}(\mathbf{y})) d\mathbf{y}, \quad \mathbb{S} = \int_S \lambda(\mathbf{y}) \mathbf{n}_y \otimes \mathbf{n}_y ds_y.$$

The techniques used are inspired from [6] where the authors consider a constant surface tension problem, but do not include the kinematic condition, (2.2.1d), on the droplet surface, thus the expansion in their case has an extra term. In [42], we show that the kinematic condition is essential for recovering G. I. Taylor's celebrated formula for the viscosity of a suspension of spherical fluid droplets in a viscous fluid. In the works of [2] and [6] one can find computations of the above tensors in the case of spherical and ellipsoidal droplet shapes.

The kinematic boundary condition is essential to obtaining the proper solution to problem (0.0.3). The use of the kinematic condition leads to the recovery of the proper effective viscosity given by Taylor's formula [57]. The results of this section rely on the work of [23] and [33] to compute the effective viscosity of the emulsion. We assume that the droplets have a spherical shape with fixed centers of mass. Since the shape of the drop has already been determined, one cannot impose the normal stress balance. However, due to the spherical shape of the drop the problem for two concentric circles can be solved explicitly. Hence, for low concentrations

$$\mu_{ijkl}^H = \mu_2 A_{ijkl} \left(1 + \frac{5\eta + 2}{2(1 + \eta)} \phi + \mathcal{O}(\phi^2) \right) \quad (0.0.4)$$

where $\eta = \mu_1/\mu_2$ and A_{ijkl} is the symmetric, trace free fourth order tensor defined by

$$A_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{kl}$$

Therefore, the effective scalar viscosity μ^* , is the viscosity of the Newtonian fluid that behaves as the mixture of arbitrarily oriented spherical droplets of viscosity μ_d , that are invariant under suspension rotation. The effective scalar viscosity is computed as the angular averaging of the tensor in (0.0.4)

$$\mu^* = \mu_0 \left(1 + \frac{5\eta + 2}{2(1 + \eta)} \phi + \mathcal{O}(\phi^2) \right) \quad (0.0.5)$$

which agrees with Taylor's result (see [57]).

Chapter three is focused on the homogenization of magnetorheological (MR) fluids. MR fluids are a suspension of non-colloidal, ferromagnetic particles in a non-magnetizable carrier fluid. They respond to an external magnetic field by a rapid, reversible change in their properties. In the absence of a magnetic field, these rigid particles align themselves along the direction of flow, however, when a magnetic field is applied each rigid particle aligns itself along the direction of the magnetic field. The rigid particles form a chain like structure along the lines of magnetic flux increasing its yield strength. This yield strength depends on the strength of the magnetic field as well as the concentration of the rigid particles [34], [21].

The formation of these aggregates means that the behavior of the fluid is non-Newtonian. In many works, the Bingham constitutive law is used as an approximation to model the response of the MR and ER fluids, particularly in shear experiments [47], [20], [22]. Although the Bingham model has proven itself useful in characterizing the behavior of MR fluids, it is not always sufficient. Recent experimental data show that true MR fluids exhibits departures from the Bingham model [61], [22].

Another member of the magnetic suspensions family are ferrofluids. Ferrofluids are stable colloidal suspensions of nanoparticles in a non-magnetizable carrier fluid. The initiation into the hydrodynamics of ferrofluids began with Neuringer and Rosensweig in 1964 [46] and by a series of works by Rosensweig and co-workers summarized in [50]. The model introduced

in [46] assumes that the magnetization is collinear with the magnetic field and has been very useful in describing quasi-stationary phenomena. This work was extended by Shliomis [54] by avoiding the collinearity assumption of the magnetization and the magnetic field and by considering the rotation of the nanoparticles with respect to the fluid they are suspended in.

The work in this chapter focuses on a suspension of rigid magnetizable particles in a Newtonian viscous fluid with an applied external magnetic field. The fluid is assumed to be electrically non-conducting. Thus, we use the quasi-static Maxwell equations coupled with the Stokes equations to capture the magnetorheological effect. For the homogenization setting of the suspension problem we define $\Omega \subset \mathbb{R}^3$, to be a bounded open set with sufficiently smooth boundary $\partial\Omega$, $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^3$ be the unit cube in \mathbb{R}^3 , and \mathbb{Z}^3 is the set of all three dimensional vectors with integer components. For every positive ϵ , let N^ϵ be the set of all points $\ell \in \mathbb{Z}^3$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with sufficiently smooth boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subset \subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon(\ell + T)$. The set T_ℓ^ϵ represents one of the solid particles suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface (see FIGURE 2).

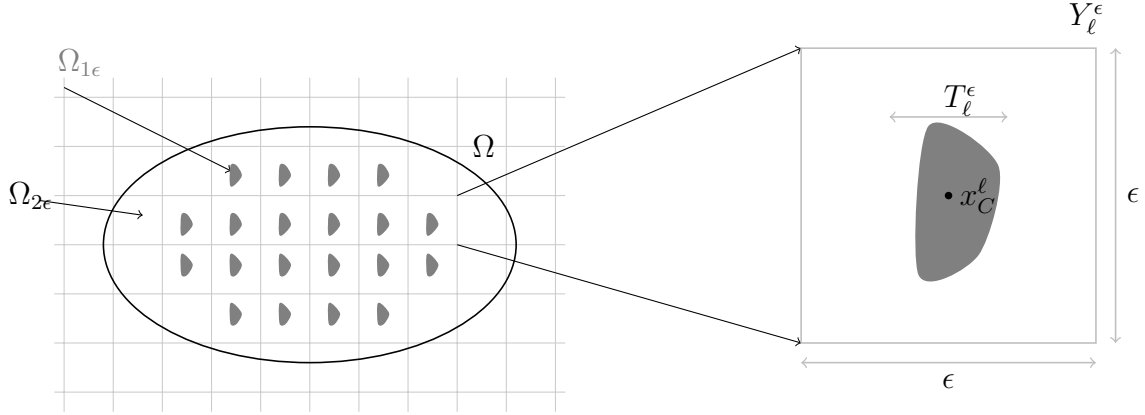


Figure 2: Schematic of the periodic suspension of rigid magnetizable particles in non-magnetizable fluid

We now define the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}},$$

where $\Omega_{1\epsilon}$ is the domain occupied by the rigid particles, and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid of viscosity ν . By \mathbf{n} we indicate the unit normal on the particle surface pointing outwards and by $[[\cdot]]$ we indicate the jump discontinuity between the fluid and the

rigid part. For low Reynolds number the description of the problem is,

$$\begin{aligned} -\operatorname{div} \sigma^\epsilon &= \mathbf{0}, \quad \text{where } \sigma^\epsilon = 2e(\mathbf{v}^\epsilon) - p^\epsilon I && \text{in } \Omega_{2\epsilon}, \\ \operatorname{div} \mathbf{v}^\epsilon &= 0, \quad \operatorname{div} \mathbf{H}^\epsilon = 0, \quad \operatorname{curl} \mathbf{H}^\epsilon = \mathbf{0} && \text{in } \Omega_{2\epsilon}, \\ e(\mathbf{v}^\epsilon) &= 0, \quad \operatorname{div} \mathbf{H}^\epsilon = 0, \quad \operatorname{curl} \mathbf{H}^\epsilon = R_m \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon && \text{in } \Omega_{1\epsilon}, \end{aligned}$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^\epsilon \rrbracket &= \mathbf{0}, \quad \llbracket \mathbf{B}^\epsilon \cdot \mathbf{n} \rrbracket = 0, \quad \llbracket \mathbf{n} \times \mathbf{H}^\epsilon \rrbracket = \mathbf{0} && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{0}, \quad \mathbf{H}^\epsilon = \mathbf{b} && \text{on } \partial\Omega, \end{aligned}$$

together with the balance of forces and torques,

$$\begin{aligned} 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n} \, ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n} \rrbracket \, ds - \alpha \int_{T_\ell^\epsilon} \mathbf{B}^\epsilon \times \operatorname{curl} \mathbf{H}^\epsilon \, d\mathbf{x}, \\ 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n} \times (\mathbf{x} - \mathbf{x}_c^\ell) \, ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n} \rrbracket \times (\mathbf{x} - \mathbf{x}_c^\ell) \, ds \\ &\quad - \alpha \int_{T_\ell^\epsilon} (\mathbf{B}^\epsilon \times \operatorname{curl} \mathbf{H}^\epsilon) \times (\mathbf{x} - \mathbf{x}_c^\ell) \, d\mathbf{x}, \end{aligned}$$

Here \mathbf{v}^ϵ represents the velocity field, p^ϵ the pressure, $e(\mathbf{v}^\epsilon)$ the strain rate, \mathbf{f} the body forces, \mathbf{n} the exterior normal to the particles, \mathbf{H}^ϵ the magnetic field, μ^ϵ is the magnetic permeability of the material, $\mu^\epsilon(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_{1\epsilon}$ and $\mu^\epsilon(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_{2\epsilon}$, η the electrical conductivity of the rigid particles, τ^ϵ is the Maxwell stress tensor present in the entire domain Ω given by $\tau_{ij}^\epsilon = \mu^\epsilon H_i^\epsilon H_j^\epsilon - \frac{1}{2} \mu^\epsilon H_k^\epsilon H_k^\epsilon \delta_{ij}$, and \mathbf{b} is an applied constant magnetic field on the exterior boundary of the domain Ω , $\alpha = \frac{\mu_2 H^2 L}{\nu V}$ is the Alfven number, and $R_m = \eta \mu_1 L V$ is the magnetic Reynolds number.

Using a two scale expansion on \mathbf{v}^ϵ , \mathbf{H}^ϵ and p^ϵ we obtain at order $\mathcal{O}(\epsilon^{-1})$ three local problems. The first problem is the contributions of the Maxwell equations,

$$\begin{aligned} \text{find } \phi^k &\in W_{per}(Y) \text{ such that} \\ \int_Y \mu \frac{\partial \phi^k}{\partial y_i} \frac{\partial v}{\partial y_i} \, d\mathbf{y} &= \int_Y \mu \frac{\partial v}{\partial y_k} \, d\mathbf{y}. \end{aligned}$$

Here, Y is unit cell, μ is the magnetic permeability in Y and

$$W_{per}(Y) = \{w \in H_{per}^1(Y) \mid \llbracket w \rrbracket = 0 \text{ on } S, \quad \tilde{w} = 0\}.$$

In addition to the contribution of Maxwell's equations we obtain a bulk velocity contribution,

$$\begin{aligned} \text{find } \chi^{m\ell} &\in \mathcal{U}_{ad} \text{ such that} \\ \int_{Y_f} 2e_{ijy}(\chi^{m\ell}) e_{ijy}(\phi - \chi^{m\ell}) \, d\mathbf{y} &= 0 \text{ for all } \phi \in \mathcal{U}_{ad}. \end{aligned}$$

where

$$\mathcal{U}_{ad} = \{\mathbf{u} \in (H_{per}^1(Y))^3 \mid \operatorname{div} \mathbf{u} = 0 \text{ in } Y_f, \quad e_{ijy}(\mathbf{u}) = -C_{ijm\ell} \text{ in } T, \\ \llbracket \mathbf{u} \rrbracket = \mathbf{0} \text{ on } S, \quad \tilde{\mathbf{u}} = \mathbf{0} \text{ in } Y\}$$

is a closed, convex, non-empty subset of $(H_{per}^1(Y))^3$. Here $C_{ijm\ell} = \frac{1}{2}(\delta_{im}\delta_{jl} + \delta_{il}\delta_{jm}) - \frac{1}{3}\delta_{ij}\delta_{m\ell}$ is a symmetric, trace free fourth order tensor. We remark that if we define $B_k^{ij} = \frac{1}{2}(y_i\delta_{jk} + y_j\delta_{ik}) - \frac{1}{3}y_k\delta_{ij}$, then $e_{ij}(\mathbf{B}^{m\ell}) = C_{ijm\ell}$.

The third problem is a bulk magnetic field contribution,

$$\text{find } \boldsymbol{\xi}^{m\ell} \in V_{per}(Y) \text{ such that} \\ \int_{Y_f} 2 e_{ijy}(\boldsymbol{\xi}^{m\ell}) e_{ijy}(\boldsymbol{\phi}) d\mathbf{y} + \int_Y A_{ij}^{m\ell} e_{ijy}(\boldsymbol{\phi}) d\mathbf{y} = 0 \text{ for all } \boldsymbol{\phi} \in V_{per}(Y).$$

where

$$V_{per}(Y) = \{\mathbf{v} \in (H_{per}^1(Y))^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } Y_f, \quad e_{ijy}(\mathbf{v}) = 0 \text{ in } T, \quad \llbracket \mathbf{v} \rrbracket = \mathbf{0} \text{ on } S, \\ \tilde{\mathbf{v}} = \mathbf{0} \text{ in } Y\}$$

is a closed subspace of $(H_{per}^1(Y))^3$.

Here $A_{i\ell}(\mathbf{y}) = \left(-\frac{\partial \phi^\ell(\mathbf{y})}{\partial y_i} + \delta_{i\ell}\right)$ and $A_{ij}^{m\ell} = \frac{1}{2}(A_{i\ell}A_{jm} + A_{j\ell}A_{im} - A_{mk}A_{\ell k}\delta_{ij})$ with the following symmetry, $A_{ij}^{m\ell} = A_{ji}^{m\ell} = A_{ij}^{\ell m}$.

Based on the local problems above we can write down the homogenized equations governing the MR fluid,

$$\begin{aligned} \frac{\partial}{\partial x_j} (\sigma_{ij}^H + \tau_{ij}^H) &= 0 & \text{in } \Omega, \\ \sigma_{ij}^H + \tau_{ij}^H &= -\bar{p}^0 \delta_{ij} + \nu_{ijm\ell} e_{m\ell x}(\mathbf{v}^0) + \beta_{ijm\ell} \tilde{H}_m^0 \tilde{H}_\ell^0, \\ \frac{\partial v_i^0}{\partial x_i} &= 0 & \text{in } \Omega, \\ \frac{\partial(\mu_{jk} \tilde{H}_k^0)}{\partial x_j} &= 0, \quad \epsilon_{ijk} \frac{\partial \tilde{H}_k^0}{\partial x_j} = R_m \epsilon_{ijk} v_j^0 \mu_{kp}^S \tilde{H}_p^0 & \text{in } \Omega, \\ v_i^0 &= 0, \quad \tilde{H}_i^0 = b_i & \text{in } \Omega. \end{aligned}$$

Here μ_{jk} , μ_{jk}^S are the effective magnetic permeabilities over the cell Y and over the rigid particle T computed with help of the Maxwell local problem, $\nu_{ijm\ell}$ and $\beta_{ijm\ell}$ constitute the effective viscosity and the effective coefficients of a ‘‘Maxwell type’’ stress given by

$$\nu_{ijm\ell} = \int_{Y_f} 2 e_{pq}(\mathbf{B}^{m\ell} + \boldsymbol{\chi}^{m\ell}) e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y}.$$

$$\beta_{ijm\ell} = \int_{Y_f} 2 e_{pq}(\boldsymbol{\xi}^{m\ell}) e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y} + \alpha \int_{Y_f} \mu A_{pq}^{m\ell} e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y} + \int_Y \mu A_{ij}^{m\ell} d\mathbf{y}.$$

The effective coefficients for the Maxwell type stress generalize the effective coefficients obtained in [27]. Below are some plots and streamlines of the solutions for the bulk velocity and magnetic field contributions,

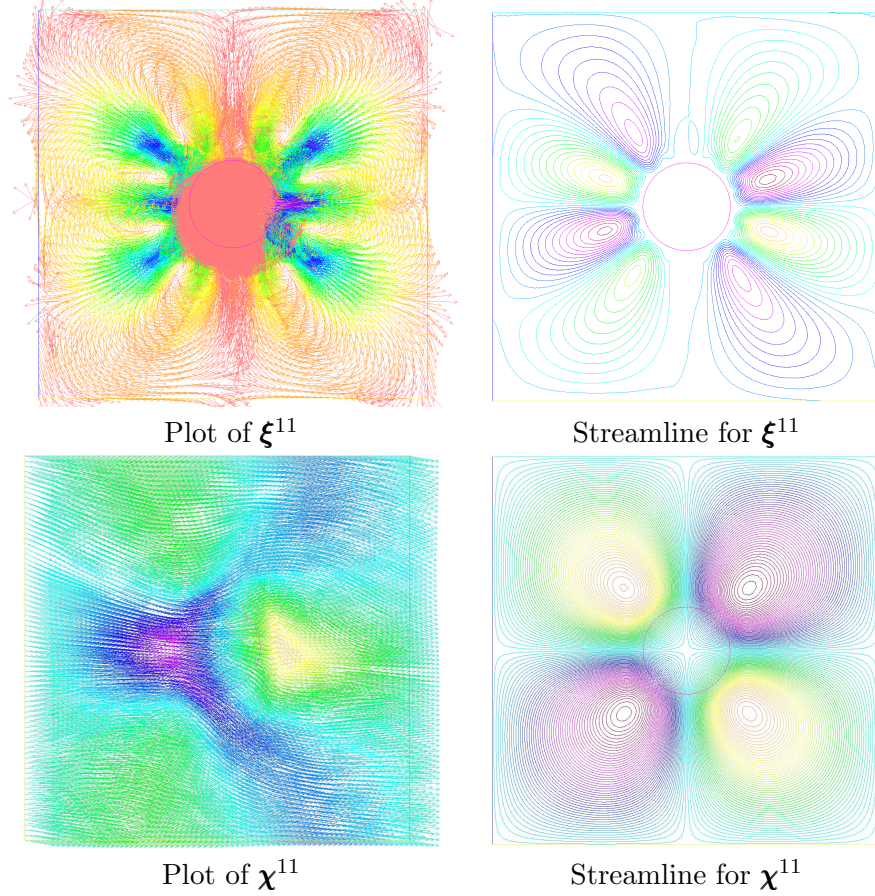
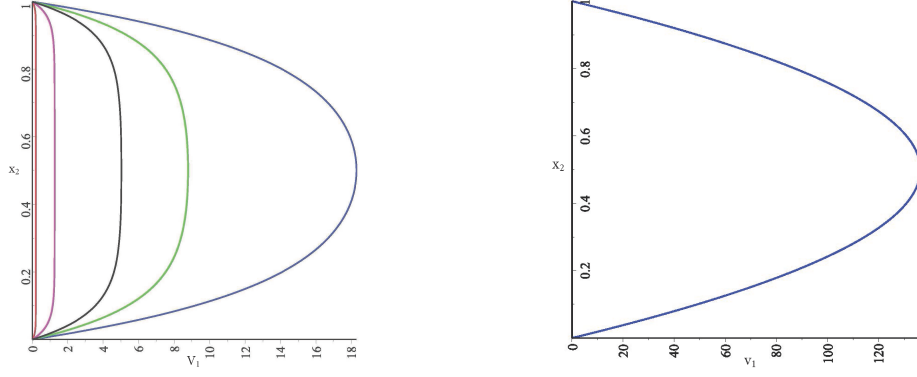


Figure 3: Plots of the local solutions for bulk velocity and bulk magnetic field contributions for magnetite nanoparticles of volume fraction $\phi = 0.07$ generated using **FreeFem++**.

For shear experiments, the response of MR fluids is often modeled using a Bingham constitutive law [47], [20], [21]. Although the Bingham constitutive law measures the response of the MR fluid quite reasonably, actual MR fluid behavior exhibits departures from the Bingham model [61],[22].



Velocity profile for MR Poiseuille flow Velocity profile for regular Poiseuille flow

Figure 4: On the left the magnetic field $B_2=0.05, 0.02, 0.01, 0.0075, 0.005$ T, respectively. On the right $B_2=0$ T.

As an application of the homogenized model, we considered a unidirectional steady flow between two infinite, parallel, stationary plates. The calculations we carried out for magnetite nanoparticles with volume fraction $\phi = 0.07$. Plotting the velocity profile for different values of magnetic field intensity, we can see that the flow does not resemble a Bingham flow when the values of the magnetic field are small (see FIGURE 4). However, as the magnetic field intensity increases the flow begins to approach a Bingham fluid flow.

The model above generalizes the model of Neuringer and Rosensweig [46] by coupling the velocity of the fluid with the magnetic field intensity. Moreover, the effective coefficients of viscosity and magnetic permeability can be computed explicitly from the local problems. One of the difficult aspects from a mathematical viewpoint is the existence of solutions to systems of MR fluids. In forthcoming work we devote our efforts in studying qualitative properties of solutions to such systems and construct appropriate test functions using the Biot-Savart law. Moreover, we show existence of a solution to the suspension system using fixed point theory.

The results in this thesis are based on the following articles:

- G. Nika and B. Vernescu, *Asymptotics for dilute emulsions with surface tension*, J. Elliptic & Parabolic Equat., **1** (2015) pp. 215–230.
- G. Nika and B. Vernescu, *Dilute emulsions with surface tension*, Quart. Appl. Math. **74** (2016), pp. 89–111.
- G. Nika and B. Vernescu, *Rate of convergence for a multi-scale model of dilute emulsions with non-uniform surface tension*, Discr. and Cont. Dyn. Syst. (in print).
- G. Nika and B. Vernescu, *Homogenization for a multi-scale model of magnetorheological suspension*, (in preparation).

Chapter 1

Asymptotics for a multi-scale model of dilute emulsions with surface effects: the case of non-convected droplets

1.1 Viscous drops in a viscous fluid

The literature on emulsions, and in particular the study of their effective properties, is vast and starts with the work on dilute emulsions by Taylor [57] who considered an emulsion formed by two newtonian, incompressible fluids, one of which is dispersed in the other, in the form of spherical droplets, with fixed centers of mass, and derived the form of its effective viscosity:

$$\mu^{eff} = \left(1 + \frac{5\mu_1 + 2\mu_2}{2(\mu_1 + \mu_2)}\phi + \mathcal{O}(\phi^2)\right)\mu_2, \quad (1.1.1)$$

in the case of droplets that have fixed centers of mass (i.e. are not convected with the flow); here μ_1 and ϕ are the viscosity and respectively the volume fraction of the droplets and μ_2 the viscosity of the continuous liquid phase. The formula generalizes the celebrated Einstein's formula for the viscosity of suspensions of spherical, rigid particles (as $\mu_1 \rightarrow \infty$), and considers the so called “zero-th order” approximation, i.e. the case when the droplets remain spherical. The “first order” approximation of the droplet deformation was considered in the case of dilute emulsions, by Schowalter, Chaffey and Brenner [53] and Frankel and Acrivos [18] who derived a non-newtonian behavior of the emulsions that exhibit “fluid memory” effects.

Another model of dilute fluid droplets, was introduced by Ammari et al. [2]. While for the limit case of rigid particles, the Einstein's formula can be recovered from this work, Taylor's formula (1.1.1) cannot be obtained, as their model, while using the equations in the Eulerian frame, imposes the boundary conditions in the Lagrangian frame. The same model is used by Bonnetier, Manceau and Triki [6] who extend to the case with surface tension, for

droplets of known curvature.

In the non-dilute case, constitutive equations for emulsions have been derived by Choi and Schowalter [12] who considered “first order” approximation of the droplet deformation. In the framework of periodic homogenization, the effective behavior of emulsions was studied by Lipton and Vernescu [33] in the case of spherical droplets that are convected with the fluid, results that extended the case of spherical droplets with fixed centers studied by Lipton and Avellaneda [32]. In the former an effective viscosity was derived consistent with the effective stress formula of Batchelor [5]. In the latter, neglecting the bubble velocity, the problem yielded a Darcy flow, since it is equivalent to a flow around fixed obstacles.

The present chapter focuses on the dilute case of droplets in the periodic homogenization framework. We consider the time-dependent, slow motion of a two-fluid dilute emulsion formed by two newtonian, incompressible fluids, one of which is dispersed in the other under the form of droplets and derive its effective behavior. We consider only the “zero-th order” problem, in which the effects of droplet deformation is not taken into account. Thus if we denote by Ω_1 the domain occupied by the droplets, of viscosity μ_1 , and by Ω_2 the domain occupied by the continuous liquid phase of viscosity μ_2 , and by S the union of the bubble surfaces (i.e. $S = \overline{\Omega}_1 \cap \overline{\Omega}_2$), the problem is described by

$$\frac{\partial \mathbf{v}}{\partial t} - \operatorname{div}(-pI + 2\mu(\mathbf{x})e(\mathbf{v})) = \mathbf{f} \quad \text{in } \Omega_1 \cup \Omega_2, \quad (1.1.2)$$

$$\operatorname{div} \mathbf{v} = 0 \quad \text{in } \Omega, \quad (1.1.3)$$

where $\mu(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_1$ and $\mu(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_2$.

The droplets are periodically distributed and the size of the period is much larger than the characteristic length of the droplets. This corresponds to a zero limit concentration of droplets. We assume, that the fluid velocity is continuous across the droplet surface and both a kinematic and a dynamic condition are imposed on the fluid interface. In addition we impose the condition that the droplets are neutrally buoyant.

The formulation of the stationary problem is discussed in section 1.2. where, for the reader’s convenience, we give details on the boundary conditions that need to be imposed on the fluid interface S , the droplets’s boundary: i. a no-slip condition, ii. a kinematic condition (that expresses the fact that droplet boundary is a material boundary) and iii. a dynamic condition (expressing the stress jump in terms of the surface tension). In addition, the balance equations for the forces and torques on each droplet needs to be imposed, condition expressing the fact that the droplets are neutrally buoyant. More details on boundary conditions can be found in the monographs of Leal [25] and Zapryanov and Tabakova [62].

In section 1.3 we formulate the periodic homogenization problem for droplets of arbitrary shapes as a variational problem. While two interesting cases can be derived: the case of convected or non-convected droplets, we further detail the case of droplets with fixed centers of mass and give its weak formulation.

Section 1.4 is dedicated to the so called “local problem”. We identify a critical size of droplets, for which the sequence of solutions loses compactness.

In section 1.5 we give the main result, the Γ -convergence of the functionals describing the periodic problem to a limit functional, which has in the critical case an extra term, the

limit in this case being a Brinkman type equation. The limit problem is of the form

$$-\operatorname{div}(-pI + 2\mu_2 e(\mathbf{v})) + \mathcal{M}\mathbf{v} = \mathbf{f} \quad \text{in } \Omega, \quad (1.1.4)$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \quad (1.1.5)$$

where \mathcal{M} is a symmetric second order tensor, that depends on the geometry of the droplets and on the two fluid viscosities. The Brinkman law obtained by Brillard [11] and Allaire [1] can be obtained from here when $\mu_1 \rightarrow +\infty$. For the problem of convected droplets, the limit corresponds to the unperturbed flow, i.e. $\mathcal{M} \equiv 0$.

The particular case of spherical droplets is considered in section 1.6 and in this case the explicit form of the tensor \mathcal{M} is found:

$$\mathcal{M}_{mk} = m\mu_2 \frac{\pi}{8} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \delta_{mk},$$

with $m = \lim_{\epsilon \rightarrow 0} \frac{a_\epsilon}{\epsilon^3}$. Let us remark here that in the case of suspensions of spherical rigid particles, the tensor reduces to

$$\mathcal{M}_{mk} = m\mu_2 \frac{3\pi}{8} \delta_{mk}.$$

The time-dependent case, is treated via Mosco convergence in section 1.7. The Appendix contains the derivation of the weak formulation and some technical results regarding the local problems.

The results in this chapter are based on Nika and Vernescu [43].

1.2 Problem statement

1.2.1 Balance of mass and momentum

Let us denote by Ω the domain occupied by the emulsion, by Ω_1 the domain occupied by the droplets, of viscosity μ_1 , and by Ω_2 the domain occupied by the continuous liquid phase of viscosity μ_2 and $\Omega = \Omega_1 \cup \overline{\Omega_2}$. The droplets are denoted by T_ℓ and their surface by S_ℓ ; the union of the bubble surfaces $S = \overline{\Omega_1} \cap \overline{\Omega_2}$. The problem is described by the balance of momentum and mass equations

$$-\operatorname{div}(-pI + 2\mu(\mathbf{x}) e(\mathbf{v})) = \mathbf{f} \quad \text{in } \Omega_1 \cup \Omega_2, \quad (1.2.1)$$

$$\operatorname{div}\mathbf{v} = 0 \quad \text{in } \Omega, \quad (1.2.2)$$

where \mathbf{v} and p represent the fluid velocity and pressure, \mathbf{f} are the body forces, and the viscosity $\mu(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_1$ and $\mu(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_2$. The stress tensor will be denoted by $\sigma = -pI + 2\mu(\mathbf{x}) e(\mathbf{v})$, where $e(\mathbf{v}) = 1/2(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ is the strain rate tensor.

1.2.2 Boundary conditions on droplet surfaces

i. A kinematic boundary condition needs to be imposed on the droplet boundary, that expresses the fact that the boundary remains an interphase boundary.

Let us assume that the shape of the droplets is given by the surface $F(t, \mathbf{x}) = 0$. Then

$$0 = \frac{dF}{dt} = \frac{\partial F}{\partial t} + \nabla_x F \cdot \mathbf{v},$$

and thus the normal velocity of the droplet boundary is given by

$$\mathbf{v} \cdot \mathbf{n} = -\frac{\frac{\partial F}{\partial t}}{|\nabla_x F|}.$$

The kinematic boundary condition on the droplet surface imposes the normal velocity of both fluids to be equal to the normal velocity of the surface:

$$\llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket = 0, \text{ and } \mathbf{v} \cdot \mathbf{n} = -\frac{\frac{\partial F}{\partial t}}{|\nabla_x F|}. \quad (1.2.3)$$

Let us now assume that, although it moves, the droplet shape does not change in time, thus $F(t, \mathbf{x}) = G(\mathbf{x}')$, where \mathbf{x}' are the coordinates of a point on the droplet surface in a moving frame, with orthonormal base $\{\mathbf{e}'_i\}$, centered at the center of mass \mathbf{x}_C^ℓ of the droplet T_ℓ . Thus

$$\mathbf{x}' = \mathbf{x} - \mathbf{x}_C^\ell \text{ and } \frac{d\mathbf{e}'_i}{dt} = A_{ij}\mathbf{e}'_j,$$

with $A = (A_{ij})$ an antisymmetric matrix. Then

$$0 = \frac{dG}{dt} = \frac{\partial G}{\partial x'_i} \frac{dx'_i}{dt} = n'_i((\mathbf{v} - \mathbf{v}_C^\ell) \cdot \mathbf{e}'_i + (\mathbf{x} - \mathbf{x}_C^\ell) \cdot (A_{ik}\mathbf{e}'_k)),$$

and thus the normal velocity of the interface S_ℓ is given by:

$$\mathbf{v} \cdot \mathbf{n} = (\mathbf{v}_C^\ell + A(\mathbf{x} - \mathbf{x}_C^\ell)) \cdot \mathbf{n}. \quad (1.2.4)$$

Thus the kinematic boundary condition (1.2.3) becomes

$$\llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket = 0, \text{ and } \mathbf{v} \cdot \mathbf{n} = (\mathbf{v}_C^\ell + A(\mathbf{x} - \mathbf{x}_C^\ell)) \cdot \mathbf{n}. \quad (1.2.5)$$

The angular velocity \mathbf{c} can be defined in \mathbb{R}^3 as $\mathbf{c} = (c_1, c_2, c_3)$ with $c_1 = -A_{23}$, $c_2 = -A_{31}$, and $c_3 = -A_{12}$ and therefore

$$\frac{d\mathbf{e}'_i}{dt} = \mathbf{c} \times \mathbf{e}'_i,$$

and the kinematic boundary condition (1.2.5) becomes

$$\llbracket \mathbf{v} \cdot \mathbf{n} \rrbracket = 0, \text{ and } \mathbf{v} \cdot \mathbf{n} = (\mathbf{v}_C^\ell + \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell)) \cdot \mathbf{n}. \quad (1.2.6)$$

Remark 1.2.1. Let us observe that the kinematic boundary condition (1.2.5) implies

$$\mathbf{v}_C^\ell = \frac{1}{|T_\ell|} \int_{T_\ell} \mathbf{v} d\mathbf{x}.$$

Indeed using the incompressibility condition we have

$$\int_{S_\ell} x_i v_j n_j ds = \int_{T_\ell} (v_j \delta_{ij} + x_i \operatorname{div} \mathbf{v}) d\mathbf{x} = \int_{T_\ell} v_i d\mathbf{x},$$

and using the interphase velocity equation (1.2.4) we obtain

$$\begin{aligned} \int_{S_\ell} x_i v_j n_j ds &= \int_{S_\ell} ((v_C^\ell)_j x_i n_j + x_i (x_l - x_{Cl}^\ell) A_{lj} n_j) ds \\ &= |T_\ell| (v_C^\ell)_i + \int_{T_\ell} (x_l - x_{Cl}^\ell) A_{li} d\mathbf{x} + \int_{T_\ell} x_i A_{li} d\mathbf{x}. \end{aligned} \quad (1.2.7)$$

The last integral above cancels because of the antisymmetry of A , and the one before last from the definition of the center of mass. In this case the kinematic condition (1.2.4) becomes

$$\mathbf{v} \cdot \mathbf{n} = \left(\frac{1}{|T_\ell|} \int_{T_\ell} \mathbf{v} d\mathbf{x} + A(\mathbf{x} - \mathbf{x}_C^\ell) \right) \cdot \mathbf{n}.$$

In the particular case of spherical droplets $\mathbf{x} - \mathbf{x}_C^\ell$ is parallel to \mathbf{n} and thus the kinematic condition (1.2.5) reduces to

$$[\![\mathbf{v} \cdot \mathbf{n}]\!] = 0, \text{ and } \mathbf{v} \cdot \mathbf{n} = \frac{1}{|T_\ell|} \left(\int_{T_\ell} \mathbf{v} d\mathbf{x} \right) \cdot \mathbf{n}. \quad (1.2.8)$$

ii. A second type of boundary condition connects the stress in each fluid at the boundary. Indeed on the droplet surface there is a stress jump $[\![\boldsymbol{\sigma} \mathbf{n}]\!] \neq 0$, and (1.2.1) is only valid in Ω_1 and Ω_2 and therefore we have

$$-\operatorname{div} \boldsymbol{\sigma} = [\![\boldsymbol{\sigma} \mathbf{n}]\!] \delta_{S_\ell} + \mathbf{f} \text{ in } \Omega, \quad (1.2.9)$$

$$\boldsymbol{\sigma} = -pI + 2\mu e(\mathbf{v}), \quad (1.2.10)$$

with δ_{S_ℓ} the Dirac measure on S_ℓ . The stress jump can be obtained from the principle that the forces on an element of interfacial area of arbitrary shape and size must be in equilibrium, because the interface is assumed to have zero thickness and thus zero mass. One can thus obtain (see APPENDIX A.4)

$$[\![\boldsymbol{\sigma} \mathbf{n}]\!] = \lambda(\nabla_s \cdot \mathbf{n})\mathbf{n} - \nabla_s \lambda,$$

where λ is the surface tension and $\nabla_s = \nabla - \mathbf{n}(\mathbf{n} \cdot \nabla)$ is the surface gradient operator. If the surface tension is uniform, the stress has only a normal jump across the interface, that is proportional to the surface tension and the mean curvature.

iii. A third type of boundary condition needs to be imposed if the droplets do not change shape. In this case the droplet surface acts as a rigid surface that needs to be in equilibrium as the viscous stresses act on it, and thus the balance of forces and torques need to be satisfied:

$$\int_{S_\ell} \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0, \text{ and } \int_{S_\ell} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0. \quad (1.2.11)$$

If the droplets are not allowed to translate, the balance of forces and torques becomes

$$\int_{S_\ell} \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds + \mathbf{F}_\ell \delta_{C\ell} = 0, \text{ and } \int_{S_\ell} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0, \quad (1.2.12)$$

where \mathbf{F}_ℓ is a pointwise force, centered at the center of mass of each droplet which keeps the droplet from translating with the fluid, but which thus gives no extra torque; $\delta_{C\ell}$ is the Dirac mass at the center of the droplet. In this case the balance of forces equation (1.2.12) can be used to determine \mathbf{F}_ℓ .

Let us observe that in the case of spherical droplets of radius R with constant surface tension $\lambda = \text{const.}$ the stress jump becomes

$$\llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket = \frac{\lambda}{R} \mathbf{n},$$

and the balance of forces on the droplet surface is automatically satisfied:

$$\int_{S_\ell} \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds = 0.$$

1.2.3 Boundary conditions on the exterior boundary

For simplicity on the exterior boundary we will consider a no-slip condition:

$$\mathbf{v} = \mathbf{0} \text{ on } \partial\Omega. \quad (1.2.13)$$

1.3 Periodic homogenization for droplets of arbitrary shape

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with sufficiently smooth boundary $\Gamma = \partial\Omega$, and let $Y = (-\frac{1}{2}, \frac{1}{2})^3$ be the unit cube in \mathbb{R}^3 . For every $\epsilon > 0$, let N^ϵ be the set of all points $\ell \in \mathbb{Z}^3$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with Lipschitz boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subset \subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon\ell + a_\epsilon T$, where $a_\epsilon \ll \epsilon$. The set T_ℓ^ϵ represents one of the droplets suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface.

We now define the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}},$$

where $\Omega_{1\epsilon}$ is the domain occupied by the droplets of viscosity μ_1 , and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid, of viscosity μ_2 . Let \mathbf{n} be the unit normal on the boundary of $\Omega_{2\epsilon}$ that points outside the domain.

The problem describing the flow of the emulsion is

$$-\operatorname{div} \sigma^\epsilon = \mathbf{f} \text{ in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \quad (1.3.1a)$$

$$\sigma^\epsilon = -p^\epsilon I + 2\mu^\epsilon e(\mathbf{v}^\epsilon), \quad (1.3.1b)$$

$$\operatorname{div} \mathbf{v}^\epsilon = 0 \text{ in } \Omega, \quad (1.3.1c)$$

with boundary conditions (see (1.2.5) and (1.2.12)) on the surface of each droplet T_ℓ^ϵ , $\ell \in N^\epsilon$:

$$[\![\mathbf{v}^\epsilon]\!] = 0 \quad \text{on } S_\ell^\epsilon, \quad (1.3.2a)$$

$$\mathbf{v}^\epsilon = \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \quad \text{on } S_\ell^\epsilon, \quad (1.3.2b)$$

$$\int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times [\![\sigma^\epsilon \mathbf{n}]\!] ds = 0, \quad (1.3.2c)$$

and, for simplicity, a zero velocity condition on the exterior boundary

$$\mathbf{v}^\epsilon = \mathbf{0} \quad \text{on } \Gamma, \quad (1.3.3)$$

where $[\![\cdot]\!]$ denotes the jump across S_ℓ^ϵ , $\boldsymbol{\omega}^{\ell,\epsilon}$ is an unknown, constant vector in \mathbb{R}^3 , \mathbf{x}_C^ℓ the position vector of the center of mass of the droplet T_ℓ^ϵ , and the viscosity μ^ϵ defined as

$$\mu^\epsilon(\mathbf{x}) = \begin{cases} \mu_1 & \text{if } x \in \Omega_{1\epsilon}, \\ \mu_2 & \text{if } x \in \Omega_{2\epsilon}. \end{cases}$$

Condition (1.3.2a) is the no slip condition at the interface of the two fluids, and (1.3.2b), (1.3.2c), and (1.3.3) follow from (1.2.5), (1.2.12) and (1.2.13).

1.4 Weak formulation

The emulsion flow problem in (1.3.1) – (1.3.3) has the equivalent variational formulation:

For any $\mathbf{f} \in L^2(\Omega)^3$, find $\mathbf{v}^\epsilon \in V^\epsilon$ such that,

$$\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} d\mathbf{x}, \text{ for any } \mathbf{w} \in V^\epsilon, \quad (1.4.1)$$

where V^ϵ is the closed subspace of $H_0^1(\Omega)^3$ given by:

$$V^\epsilon = \left\{ \mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \text{ on } S_\ell^\epsilon, \quad \boldsymbol{\omega}^{\ell,\epsilon} \in \mathbb{R}^3 \right\}.$$

A weak solution to (1.3.1) – (1.3.3) is any \mathbf{v}^ϵ that satisfies (1.4.1). Conversely, (1.3.1) – (1.3.3) can be obtained from (1.4.1) in the sense of distributions; the details are presented in the appendix A.1. The existence and uniqueness of a weak solution of the emulsion flow problem follows from the Lax-Milgram lemma.

Furthermore, any \mathbf{v}^ϵ solution to (1.4.1) is the unique solution of the problem:

$$\begin{cases} \text{Find } \mathbf{v}^\epsilon \in H_0^1(\Omega)^3 \text{ such that,} \\ J^\epsilon(\mathbf{v}^\epsilon) = \min_{\mathbf{u} \in H_0^1(\Omega)^3} J^\epsilon(\mathbf{u}), \end{cases} \quad (1.4.2)$$

where

$$J^\epsilon(\mathbf{u}) = \int_{\Omega} \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}), \quad (1.4.3)$$

and I_S represents the indicator function of the set S , defined by

$$I_S(s) = \begin{cases} 0 & \text{if } s \in S \\ +\infty & \text{if } s \notin S \end{cases}$$

We are interested in studying the Γ -convergence of the sequence $\{J^\epsilon\}$, when $\epsilon \rightarrow 0$.

1.5 The local problem

Let us consider the local problem for a reference cell (see Figure 1), $Y_\ell^\epsilon := \epsilon(\ell + Y)$ for some $\ell \in N^\epsilon$.

$$\begin{aligned} -\operatorname{div} \sigma^{k\epsilon} &= \mathbf{0} && \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\ \sigma^{k\epsilon} &= -q^{k\epsilon} I + 2\mu e(\mathbf{w}^{k\epsilon}), \\ \operatorname{div} \mathbf{w}^{k\epsilon} &= 0 && \text{in } B_\ell^\epsilon, \\ \llbracket \mathbf{w}^{k\epsilon} \rrbracket &= 0 && \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} &= \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{y} - \mathbf{y}_C^\ell) && \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} &= \mathbf{e}_k && \text{on } \partial B_\ell^\epsilon, \\ \int_{S_\ell^\epsilon} (\mathbf{y} - \mathbf{y}_C^\ell) \times \llbracket \sigma^{k\epsilon} \mathbf{n} \rrbracket ds &= 0, \end{aligned} \quad (1.5.1)$$

where B_ℓ^ϵ is the ball with center the center of cell $\ell \in N^\epsilon$ and radius $\epsilon/2$, \mathbf{e}_k is the k^{th} - unit vector of the cartesian base and $\mu = \mu_1$ in T_ℓ^ϵ and $\mu = \mu_2$ in $B_\ell^\epsilon - \overline{T_\ell^\epsilon}$.

Define, $\hat{\mathbf{w}}^{k\epsilon} = \mathbf{w}^{k\epsilon} - \mathbf{e}_k$. Then, (1.5.1) becomes

$$\begin{aligned}
-\operatorname{div} \hat{\boldsymbol{\sigma}}^{k\epsilon} &= \mathbf{0} && \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\
\hat{\boldsymbol{\sigma}}^{k\epsilon} &= -q^{k\epsilon} I + 2\mu e(\hat{\mathbf{w}}^{k\epsilon}), \\
\operatorname{div} \hat{\mathbf{w}}^{k\epsilon} &= 0 && \text{in } B_\ell^\epsilon, \\
\llbracket \hat{\mathbf{w}}^{k\epsilon} \rrbracket &= \mathbf{0} && \text{on } S_\ell^\epsilon, \\
\hat{\mathbf{w}}^{k\epsilon} &= -\mathbf{e}_k + \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{y} - \mathbf{y}_C^\ell) && \text{on } S_\ell^\epsilon, \\
\hat{\mathbf{w}}^{k\epsilon} &= \mathbf{0} && \text{on } \partial B_\ell^\epsilon, \\
\int_{S_\ell^\epsilon} (\mathbf{y} - \mathbf{y}_C^\ell) \times \llbracket \hat{\boldsymbol{\sigma}}^{k\epsilon} \mathbf{n} \rrbracket ds &= 0.
\end{aligned}$$

Applying a change of variable we get $\hat{\mathbf{w}}^{k\epsilon}(a_\epsilon \mathbf{x})$ and $a_\epsilon q^{k\epsilon}(a_\epsilon \mathbf{x})$ are solutions for a problem of type (A.2.1), where $\boldsymbol{\chi}^{k\frac{\epsilon}{a_\epsilon}} = \hat{\mathbf{w}}^{k\epsilon}(a_\epsilon \mathbf{x})$ and $\eta^{k\frac{\epsilon}{a_\epsilon}} = a_\epsilon q^{k\epsilon}(a_\epsilon \mathbf{x})$. Hence, using our results in appendix A.2, there exists a unique solution to (1.5.1)

1.5.1 Properties of the local solution

LEMMA 1.5.1. *The solution $(\mathbf{w}^{k\epsilon}, q^{k\epsilon})$ of (2.4.1) has the following properties:*

1. *if $a_\epsilon = o(\epsilon^3)$ then $\mathbf{w}^{k\epsilon} \rightarrow \mathbf{e}_k$ in $H^1(\Omega)^3$, $q^{k\epsilon} \rightarrow 0$ in $L^2(\Omega)$.*
2. *if $a_\epsilon = \mathcal{O}(\epsilon^3)$ then $\mathbf{w}^{k\epsilon} \rightharpoonup \mathbf{e}_k$ in $H^1(\Omega)^3$, $q^{k\epsilon} \rightharpoonup 0$ in $L^2(\Omega)$.*

Proof. First, we extend $\mathbf{w}^{k\epsilon}$ by periodicity to all of \mathbb{R}^3 . Since the number of microscopic cells, Y_ℓ^ϵ , included in $|\Omega|$ is equivalent to $|\Omega|/\epsilon^3$ we have,

$$\int_{\Omega} \mu e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \simeq \frac{|\Omega|}{\epsilon^3} \int_{B_\ell^\epsilon} \mu e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} = |\Omega| \frac{a_\epsilon}{\epsilon^3} \int_{B_{\frac{\epsilon}{a_\epsilon}}^\epsilon} \mu e(\boldsymbol{\chi}^{k\frac{\epsilon}{a_\epsilon}}) : e(\boldsymbol{\chi}^{k\frac{\epsilon}{a_\epsilon}}) d\mathbf{x}$$

From REMARK A.2.4 in appendix A.2, the corresponding limit of the last term above exists as $\epsilon \rightarrow 0$. Hence, there exists a positive constant C (independent of ϵ), such that

$$\int_{\Omega} e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \leq C.$$

Furthermore, by Korn's inequality on B_ℓ^ϵ and the fact $e(\mathbf{e}_k) = 0$, we note that:

$$\|\mathbf{w}^{k\epsilon} - \mathbf{e}_k\|_{H^1(\Omega)^3} \leq \sum_{\ell} \|\mathbf{w}^{k\epsilon} - \mathbf{e}_k\|_{H^1(B_\ell^\epsilon)} \leq \sum_{\ell} C \int_{B_\ell^\epsilon} e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \leq C.$$

Therefore, we get that $\|\mathbf{w}^{k\epsilon}\|_{H^1(\Omega)^3} < C$. Hence, by taking a subsequence still denoted by $\mathbf{w}^{k\epsilon}$ we have

$$\mathbf{w}^{k\epsilon} \rightharpoonup \mathbf{w} \quad \text{weakly in } H^1(\Omega)^3.$$

Since $\chi_{\cup_{\ell \in N^\epsilon} Y_\ell^\epsilon \setminus B_\ell^\epsilon}$ converges in the weak topology of $L^2(\Omega)^3$ to the non-zero constant $|\Omega|(1 - \pi/6)$ and $\mathbf{w}^{k\epsilon} = \mathbf{e}_k$ on $Y_\ell^\epsilon \setminus B_\ell^\epsilon$, we get that $\mathbf{w} = \mathbf{e}_k$. \square

1.6 Convergence of the energies

Using (A.2.5), let us define the energy functional $E^\epsilon: H_0^1(\Omega)^3 \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$E^\epsilon(\mathbf{u}) = \int_{\Omega} \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) \, d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}). \quad (1.6.1)$$

Our goal is to show that the sequence $(E^\epsilon)_\epsilon$ Γ -converges to $E: H_0^1(\Omega)^3 \mapsto \mathbb{R} \cup \{+\infty\}$ in the weak topology of $H_0^1(\Omega)^3$ where

$$E(\mathbf{u}) = \int_{\Omega} \mu_2 e(\mathbf{u}) : e(\mathbf{u}) \, d\mathbf{x} + \sum_{k,m=1}^3 \mathcal{M}_{mk} \int_{\Omega} u_m u_k \, d\mathbf{x} + I_V(\mathbf{u}), \quad (1.6.2)$$

with V the closed subspace of $H_0^1(\Omega)^3$ defined by

$$V = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0\},$$

and $\mathcal{M} = (\mathcal{M}_{mk})_{mk}$ the positive definite, symmetric matrix defined by

$$\mathcal{M}_{mk} = m \lim_{\epsilon \rightarrow 0} \int_{B_\ell^{\frac{\epsilon}{a_\epsilon}}} \mu e(\chi^{m \frac{\epsilon}{a_\epsilon}}) : e(\chi^{k \frac{\epsilon}{a_\epsilon}}) \, d\mathbf{x}, \quad (1.6.3)$$

and

$$m = \lim_{\epsilon \rightarrow 0} \frac{a_\epsilon}{\epsilon^3}.$$

Usually the most difficult aspect of Γ -convergence is the construction of the so called recovery sequence. For the case of rigid particles suspended in a Stokes system, such sequences were first constructed by Marchenko and Khruslov [35] (see also [11]) and later extended by Vernescu [60] to the case of fluid in particles. The construction of such sequences is done explicitly through the use of the following lemmas:

LEMMA 1.6.1. *Let $r > 0$ and $\mathbf{u} \in L^2(B(r))^3$, $\operatorname{div} \mathbf{u} = 0$ in $B(r)$. Then there exists $\tilde{\mathbf{u}} \in H^1(B(r))^3$ such that*

$$\operatorname{div} \tilde{\mathbf{u}} = 0, \quad \operatorname{curl} \tilde{\mathbf{u}} = \mathbf{u} \text{ in } B(r), \quad \tilde{\mathbf{u}} \cdot \mathbf{n} = 0 \text{ on } \partial B(r),$$

$$\|\tilde{\mathbf{u}}\|_{L^2(B(r))^3} \leq C r \|\mathbf{u}\|_{L^2(B(r))^3}, \quad \|\nabla \tilde{\mathbf{u}}\|_{L^2(B(r))^{3 \times 3}} \leq C \|\mathbf{u}\|_{L^2(B(r))^3}.$$

LEMMA 1.6.2. Let $\tilde{\mathbf{v}}$ be the function associated with $\mathbf{v}(\mathbf{x}) - \mathbf{v}(\mathbf{x}_c)$ then

$$\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon \rightarrow \mathbf{0} \text{ strongly in } H^1(\Omega)^3,$$

where ϕ_ϵ is a cut-off function with $\operatorname{supp}(\phi_\epsilon) \subset B(2a_\epsilon)$ and $\phi_\epsilon(\mathbf{x}) = 1$ on T^ϵ .

THEOREM 1.6.3. The sequence of functionals

$$E^\epsilon(\mathbf{u}) = \int_{\Omega} \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + I_{V^\epsilon}(\mathbf{u})$$

Γ -converges in the weak topology of $H_0^1(\Omega)^3$ to the functional

$$E(\mathbf{u}) = \int_{\Omega} \mu_2 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + \sum_{k,m=1}^3 \mathcal{M}_{mk} \int_{\Omega} u_m u_k d\mathbf{x} + I_V(\mathbf{u}).$$

Proof. We first remark that for every $\mathbf{u} \in H_0^1(\Omega)^3$ which is not divergence-free in Ω then one derives that

$$\Gamma - \liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}) = \Gamma - \limsup_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}) = +\infty.$$

Hence, we only have to deal with divergence-free functions. Specifically, we have to prove the following two assertions

- (a) For all $\mathbf{v}^0 \in V$ there exists a $\mathbf{v}^\epsilon \in V^\epsilon$, $\mathbf{v}^\epsilon \rightharpoonup \mathbf{v}^0$ in $H_0^1(\Omega)^3$ such that $\lim_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{v}^\epsilon) = E(\mathbf{v}^0)$,
- (b) For all $\mathbf{u}^0 \in V$, for all $\mathbf{u}^\epsilon \in V^\epsilon$, $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}^0$ in $H_0^1(\Omega)^3$ such that $\liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u}^0)$.

Part (a). Let $\mathbf{v}^0 \in C_0^\infty(\Omega)^3$ such that $\operatorname{div} \mathbf{v}^0 = 0$. Define the sequence \mathbf{v}^ϵ in the following way (see [11], [60]):

$$\mathbf{v}^\epsilon(\mathbf{x}) = \begin{cases} \mathbf{v}^0(\mathbf{x}) & \text{in } Y_\ell^\epsilon - B_\ell^\epsilon, \\ \mathbf{v}^0(\mathbf{x}) + (\mathbf{w}^{k\epsilon}(\mathbf{x}) - \mathbf{e}_k) v_k^0(\mathbf{x}_C^\ell) - \operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell} & \text{in } B_\ell^\epsilon - T_\ell^\epsilon, \\ v_k^0(\mathbf{x}_C^\ell) \mathbf{w}^{k\epsilon}(\mathbf{x}) & \text{in } T_\ell^\epsilon, \end{cases} \quad (1.6.4)$$

where $\tilde{\mathbf{v}}_{\epsilon\ell}$ is the vector valued function associated with $\mathbf{v}^0(\mathbf{x}) - \mathbf{v}^0(\mathbf{x}_C^\ell)$ constructed by LEMMA 1.6.1 and LEMMA 1.6.2.

One can now verify, that the sequence \mathbf{v}^ϵ is divergence free, belongs in $H_0^1(\Omega)^3$, $\mathbf{v}^\epsilon = \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell)$ on S_ℓ^ϵ and $\mathbf{v}^\epsilon \rightharpoonup \mathbf{v}^0$ in $H_0^1(\Omega)^3$. Hence, computing $E^\epsilon(\mathbf{v}^\epsilon)$ we obtain:

$$\begin{aligned}
E^\epsilon(\mathbf{v}^\epsilon) &= \sum_{\ell \in N^\epsilon} \int_{Y_\ell^\epsilon - T_\ell^\epsilon} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon} \mu v_k^0(\mathbf{x}_C^\ell) v_m^0(\mathbf{x}_C^\ell) e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{m\epsilon}) d\mathbf{x} \\
&+ \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2 \mu_2 v_k^0(\mathbf{x}_C^\ell) e(\mathbf{v}^0) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} - \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2 \mu_2 e(\mathbf{v}^0) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) d\mathbf{x} \\
&- \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2 \mu_2 v_k^0(\mathbf{x}_C^\ell) e(\mathbf{w}^{k\epsilon}) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) d\mathbf{x} \\
&+ \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} \mu_2 e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) d\mathbf{x} \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon} \mu v_k^0(\mathbf{x}_C^\ell) v_m^0(\mathbf{x}_C^\ell) e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{m\epsilon}) d\mathbf{x} + o(1) \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \left(\sum_{\ell \in N^\epsilon} v_k^0(\mathbf{x}_C^\ell) v_m^0(\mathbf{x}_C^\ell) \epsilon^3 \right) \frac{1}{\epsilon^3} \int_{B_\ell^\epsilon} \mu e(\mathbf{w}^{k\epsilon}) : e(\mathbf{w}^{m\epsilon}) d\mathbf{x} + o(1) \\
&= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \left(\sum_{\ell \in N^\epsilon} v_k^0(\mathbf{x}_C^\ell) v_m^0(\mathbf{x}_C^\ell) \epsilon^3 \right) \frac{a_\epsilon}{\epsilon^3} \int_{B_\ell^\epsilon} \mu e(\chi^{k \frac{\epsilon}{a_\epsilon}}) : e(\chi^{m \frac{\epsilon}{a_\epsilon}}) d\mathbf{x} + o(1).
\end{aligned}$$

By the smoothness of \mathbf{v}^0 and by LEMMA 1.5.1 we get

$$\lim_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{v}^\epsilon) = \int_{\Omega} \mu_2 e(\mathbf{v}) : e(\mathbf{v}) d\mathbf{x} + \sum_{k,m=1}^3 \mathcal{M}_{mk} \int_{\Omega} v_m v_k d\mathbf{x}.$$

For an arbitrary $\mathbf{v}^0 \in H_0^1(\Omega)^3$ we use a classical diagonalization argument to pass to the limit.

Part (b). Let $\mathbf{u}^\epsilon, \mathbf{u}^0 \in H_0^1(\Omega)^3$ such that $\operatorname{div} \mathbf{u}^\epsilon = \operatorname{div} \mathbf{u}^0 = 0$, $\mathbf{u}^\epsilon = \boldsymbol{\omega}^{\epsilon,\ell} \times (\mathbf{x} - \mathbf{x}_C^\ell)$ on S_ℓ^ϵ , and $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}^0$ in $H^1(\Omega)^3$. Let $\mathbf{v}^0 \in C_0^\infty(\Omega)^3$ such that $\operatorname{div} \mathbf{v}^0 = 0$ and define the sequence \mathbf{v}^ϵ as before. Using the convexity of E^ϵ we apply the sub-differential inequality to obtain,

$$E^\epsilon(\mathbf{u}^\epsilon) \geq E^\epsilon(\mathbf{v}^\epsilon) + \int_{\Omega} 2 \mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{u}^\epsilon - \mathbf{v}^\epsilon) d\mathbf{x}. \quad (1.6.5)$$

In order to pass to the limit in the last term we make use of LEMMA A.3.2 and obtain the following bound for the stress

$$\|\sigma^{k\epsilon}\|_{L^2(\partial B_\ell^\epsilon)^{3 \times 3}} \leq C \epsilon^3$$

Therefore, the last term of (2.5.3) becomes

$$\begin{aligned}
& \int_{\Omega} \mu_{\epsilon} e(\mathbf{v}^{\epsilon}) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} = \sum_{\ell \in N^{\epsilon}} \int_{Y_{\ell}^{\epsilon} - B_{\ell}^{\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} \\
& + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 e(\mathbf{v}^0) : e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 v_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} \\
& - \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon} - T_{\ell}^{\epsilon}} \mu_2 e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell} \phi_{\epsilon\ell}) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{T_{\ell}^{\epsilon}} \mu_1 u_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} = \\
& = \int_{\Omega \setminus \Omega_{1\epsilon}} \mu_2 e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^{\epsilon}} \int_{B_{\ell}^{\epsilon}} \mu v_k^0(\mathbf{x}_C^{\ell}) e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{w}^{k\epsilon}) d\mathbf{x} + o(1).
\end{aligned}$$

We re-write the last term as follows

$$\begin{aligned}
\frac{1}{2} \sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \int_{B_{\ell}^{\epsilon}} \sigma^{k\epsilon} : \operatorname{grad}(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} &= \frac{1}{2} \sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \int_{\partial B_{\ell}^{\epsilon}} \sigma^{k\epsilon} \mathbf{n} \cdot (\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) d\mathbf{x} \\
&\geq -C \left(\sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \epsilon^3 \right) \|\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}\|_{L^2(\Omega)^3}.
\end{aligned}$$

Putting everything together we get,

$$\begin{aligned}
E^{\epsilon}(\mathbf{u}^{\epsilon}) &\geq E^{\epsilon}(\mathbf{v}^{\epsilon}) + \int_{\Omega \setminus \Omega_{1\epsilon}} 2 \mu_2 e(\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}) : e(\mathbf{v}^0) d\mathbf{x} \\
&- C \left(\sum_{\ell \in N^{\epsilon}} v_k^0(\mathbf{x}_C^{\ell}) \epsilon^3 \right) \|\mathbf{u}^{\epsilon} - \mathbf{v}^{\epsilon}\|_{L^2(\Omega)^3}.
\end{aligned}$$

Passing to the limit first as $\epsilon \rightarrow 0$ then using a diagonalization argument to make $\mathbf{v}^0 \rightarrow \mathbf{u}^0$ in the strong topology of $H_0^1(\Omega)^3$ we get

$$\liminf_{\epsilon \rightarrow 0} E^{\epsilon}(\mathbf{u}^{\epsilon}) \geq E(\mathbf{u}^0).$$

□

COROLLARY 1.6.4. *The sequence $\{\mathbf{v}^{\epsilon}\}$ of solutions to (2.3.2) is weakly convergent in the weak topology of $H_0^1(\Omega)^3$ to \mathbf{v} solution to*

$$\begin{cases} \text{Find } \mathbf{v} \in H_0^1(\Omega)^3 \text{ such that,} \\ J(\mathbf{v}) = \min_{\mathbf{u} \in H_0^1(\Omega)^3} J(\mathbf{u}), \end{cases} \quad (1.6.6)$$

where

$$J(\mathbf{u}) = \int_{\Omega} \mu_2 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + \int_{\Omega} \mathbf{u}^{\top} \mathcal{M} \mathbf{u} d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_V(\mathbf{u}), \quad (1.6.7)$$

and

$$V = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}.$$

The Brinkman law obtained here takes into account the geometry of the droplets and their viscosity. For $\mu_1 \rightarrow \infty$ one can show that the result yields the Brinkman law for rigid particles obtained in [1] and [11].

1.7 Spherical droplets

In this section we determine the matrix $\mathcal{M} = (\mathcal{M}_{mk})_{mk}$ in (1.6.3) for the case of spherical droplets. In accordance with Section 2.2 the emulsion problem simplifies in the following way:

$$\begin{aligned} -\operatorname{div} \sigma^\epsilon &= \mathbf{f} && \text{in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \\ \sigma^\epsilon &= -p^\epsilon I + 2\mu^\epsilon e(\mathbf{v}^\epsilon), \\ \operatorname{div} \mathbf{v}^\epsilon &= 0 && \text{in } \Omega, \\ \llbracket \sigma^\epsilon \mathbf{n} \rrbracket &= (\llbracket \sigma^\epsilon \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon \cdot \mathbf{n} &= 0 && \text{on } S_\ell^\epsilon, \\ \llbracket \mathbf{v}^\epsilon \rrbracket &= \mathbf{0} && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{0} && \text{on } \Gamma. \end{aligned} \tag{1.7.1}$$

The balance of forces and torque is given by (1.2.12). The balance of torque is automatically satisfied in this case, while the balance of forces we do not write down since it is de-coupled from problem 1.7.1. Moreover, the variational formulation is the same as in (1.4.1) where V^ϵ simplifies to

$$V^\epsilon = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } S_\ell^\epsilon\}.$$

1.7.1 The local problem for spheres

Let us consider the local transmission problem for cell Y_ℓ^ϵ for some $\ell \in N^\epsilon$

$$\begin{aligned} -\operatorname{div} \sigma^{k\epsilon} &= 0 && \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\ \sigma^{k\epsilon} &= -p^\epsilon I + 2\mu e(\mathbf{w}^{k\epsilon}), \\ \operatorname{div} \mathbf{w}^{k\epsilon} &= 0 && \text{in } B_\ell^\epsilon, \\ \llbracket \sigma^{k\epsilon} \mathbf{n} \rrbracket &= (\llbracket \sigma^{k\epsilon} \mathbf{n} \rrbracket \cdot \mathbf{n}) \mathbf{n} && \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} \cdot \mathbf{n} &= 0 && \text{on } S_\ell^\epsilon, \\ \llbracket \mathbf{w}^{k\epsilon} \rrbracket &= \mathbf{0} && \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} &= \mathbf{e}_k && \text{on } \partial B_\ell^\epsilon. \end{aligned} \tag{1.7.2}$$

Problem (1.7.2) admits an explicit solution. Indeed due to the spherical symmetry we look for a solution in spherical coordinates (r, θ, ϕ) with θ being the angle between \mathbf{x} and \mathbf{e}_k and with no dependence on ϕ :

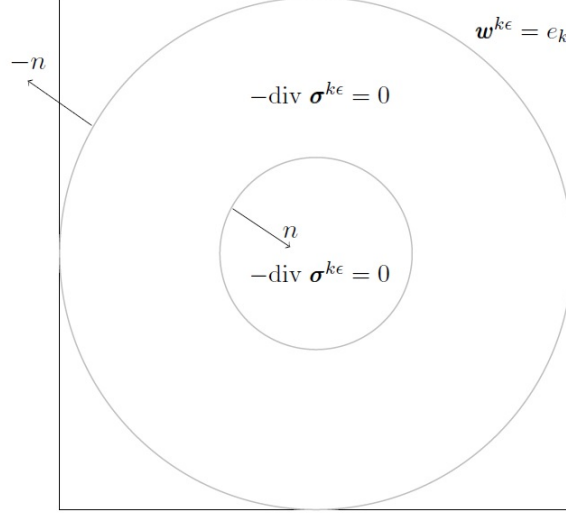


Figure 1.1: Cell problem for two concentric spheres

$$\begin{aligned} \mathbf{w}^{k\epsilon} &= f(r) \cos(\theta) \mathbf{e}_r + g(r) \sin(\theta) \mathbf{e}_\theta, \\ q^{k\epsilon} &= h(r) \cos(\theta). \end{aligned} \tag{1.7.3}$$

Here $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ denote the unit vectors in spherical coordinates. Substituting (1.7.3) in (1.7.2) we obtain the following system of differential equations:

$$\begin{aligned} \mu \left(f'' + \frac{2}{r} f' - \frac{4}{r^2} (f + g) \right) + h' &= 0, \\ \mu \left(g'' + \frac{2}{r} g' - \frac{2}{r^2} (f + g) \right) - \frac{1}{4} h &= 0, \\ f' + \frac{2}{r} (f + g) &= 0, \end{aligned} \tag{1.7.4}$$

in each domain $B_\ell^\epsilon - \overline{T}_\ell^\epsilon$ and T_ℓ^ϵ . By eliminating g and h we obtain an Euler equation for f :

$$\frac{r^2}{2} f^{(4)} + 4r f^{(3)} + r f'' - \frac{4}{r} f' = 0. \tag{1.7.5}$$

Thus the solution for (1.7.2) is given by:

$$\begin{aligned}
f(r) &= C_1 r^2 + C_2 + \frac{C_3}{r} + \frac{C_4}{r^3}, \\
g(r) &= -2 C_1 r^2 - C_2 - \frac{C_3}{2r} + \frac{C_4}{2r^3}, \\
h(r) &= -\mu \left(10 r C_1 + \frac{C_3}{r^2} \right),
\end{aligned} \tag{1.7.6}$$

inside B_ℓ^ϵ and similar solutions, with different constants say K_1, K_2, K_3, K_4 , for $B_\ell^\epsilon - \bar{T}_\ell^\epsilon$. Also, by requiring $\mathbf{w}^{k\epsilon} \in L^2(B_\ell^\epsilon)^3$, $q^{k\epsilon} \in L^2(B_\ell^\epsilon)$ we obtain $C_3 = C_4 = 0$. Furthermore, the boundary conditions yield the system:

$$\begin{aligned}
C_1 a_\epsilon^2 + C_2 &= 0, \\
K_1 a_\epsilon^2 + K_2 + \frac{K_3}{a_\epsilon} + \frac{K_4}{a_\epsilon^3} &= 0, \\
-2 C_1 a_\epsilon^2 - C_2 &= -2 K_1 a_\epsilon^2 - K_2 - \frac{K_3}{2a_\epsilon} + \frac{K_4}{2a_\epsilon^3}, \\
\mu_1 C_1 a_\epsilon &= \mu_2 \left(K_1 + \frac{K_4}{a_\epsilon^4} \right), \\
K_1 \epsilon^2 + K_2 + \frac{K_3}{\epsilon} + \frac{K_4}{\epsilon^4} &= 1, \\
-2 K_1 \epsilon^2 - K_2 - \frac{K_3}{2\epsilon} + \frac{K_4}{2\epsilon^4} &= -1.
\end{aligned} \tag{1.7.7}$$

that determine uniquely C_1, C_2, K_1, K_2, K_3 and K_4 . For instance,

$$K_1 = \frac{(3 \mu_1 a_\epsilon^2 - \epsilon^2 (3 \mu_1 + 2 \mu_2)) a_\epsilon \epsilon}{(a_\epsilon - \epsilon)^3 (4 a_\epsilon^3 (\mu_1 - \mu_2) + 3 a_\epsilon^2 \epsilon (\mu_1 - 2 \mu_2) - 3 a_\epsilon \epsilon^2 (\mu_1 + 2 \mu_2) - 4 \epsilon^3 (\mu_1 + \mu_2))}$$

We will next study the behaviour of $\mathbf{w}^{k\epsilon}, q^{k\epsilon}$ as $\epsilon \rightarrow 0$.

LEMMA 1.7.1. *The solution $(\mathbf{w}^{k\epsilon}, q^{k\epsilon})$ of (1.7.2) has the following properties:*

1. *if $a_\epsilon = o(\epsilon^3)$ then $\mathbf{w}^{k\epsilon} \rightarrow \mathbf{e}_k$ in $H^1(\Omega)^3$, $q^{k\epsilon} \rightarrow 0$ in $L^2(\Omega)$*
2. *if $a_\epsilon = \mathcal{O}(\epsilon^3)$ then $\mathbf{w}^{k\epsilon} \rightharpoonup \mathbf{e}_k$ in $H^1(\Omega)^3$, $q^{k\epsilon} \rightharpoonup 0$ in $L^2(\Omega)$, and there exists $\gamma \in H^{-1}(\Omega)$ such that*

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I), \phi \mathbf{u}^\epsilon \rangle \rightarrow \langle \gamma \mathbf{e}_k, \phi \mathbf{u} \rangle,$$

for any $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $H_0^1(\Omega)^3$ with $\mathbf{u}^\epsilon \cdot \mathbf{n} = 0$ on S_ℓ^ϵ , and for any $\phi \in C_0^\infty(\Omega)$. Moreover, if $a_\epsilon = m\epsilon^3$ then

$$\gamma = -m \frac{\pi}{4} \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}.$$

Proof. Take $\boldsymbol{\psi} \in H_0^1(\Omega)^3$ with $\boldsymbol{\psi} \cdot \mathbf{n} = 0$ on S_ℓ^ϵ . From (1.7.2) one gets:

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) , \boldsymbol{\psi} \rangle = \int_{\partial B^\epsilon} (\mu_2 \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) \boldsymbol{\psi} \cdot \mathbf{n} ds, \quad (1.7.8)$$

where $\partial B^\epsilon = \cup_{\ell \in N^\epsilon} \partial B_\ell^\epsilon$ and the right-hand side can be estimated since

$$\begin{aligned} (\mu_2 \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) \mathbf{e}_r &= (-h(\epsilon) + \mu_2 f'(\epsilon))(\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r + \mu_2 g'(\epsilon)(\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta \\ &= F(\epsilon)(\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r + G(\epsilon)(\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta. \end{aligned} \quad (1.7.9)$$

Therefore (1.7.9) becomes:

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) , \boldsymbol{\psi} \rangle = F(\epsilon) \int_{\partial B^\epsilon} (\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r \cdot \boldsymbol{\psi} ds + G(\epsilon) \int_{\partial B^\epsilon} (\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta \cdot \boldsymbol{\psi} ds. \quad (1.7.10)$$

On the other hand one has (see [13], [1]) the following convergences:

$$\sum_{\ell \in N^\epsilon} \epsilon \delta_\ell^\epsilon \mathbf{e}_k \rightarrow \frac{S_3}{2^3} \mathbf{e}_k, \text{ strongly in } W_{loc}^{-1,\infty}(\mathbb{R}^3), \quad (1.7.11)$$

$$\sum_{\ell \in N^\epsilon} \epsilon \delta_\ell^\epsilon (\mathbf{e}_k \cdot \mathbf{e}_r) \mathbf{e}_r \rightarrow \frac{1}{3} \frac{S_3}{2^3} \mathbf{e}_k, \text{ strongly in } W_{loc}^{-1,\infty}(\mathbb{R}^3), \quad (1.7.12)$$

$$\sum_{\ell \in N^\epsilon} \epsilon \delta_\ell^\epsilon (\mathbf{e}_k \cdot \mathbf{e}_\theta) \mathbf{e}_\theta \rightarrow \frac{2}{3} \frac{S_3}{2^3} \mathbf{e}_k, \text{ strongly in } W_{loc}^{-1,\infty}(\mathbb{R}^3), \quad (1.7.13)$$

where δ_ℓ^ϵ is the measure supported on ∂B^ϵ and S_3 is the surface of the unit sphere. Thus in order to pass the limit in (1.7.10) we have to compute the limit of $F(\epsilon)/\epsilon$ and $G(\epsilon)/\epsilon$. Therefore, we have

$$\frac{F(\epsilon)}{\epsilon} = \frac{-h(\epsilon) + \mu_2 f'(\epsilon)}{\epsilon} = \mu_2 \left(10 K_1 + \frac{K_3}{\epsilon^3} \right) + \mu_2 \left(2 K_1 - \frac{K_3}{\epsilon^3} - 3 \frac{K_4}{\epsilon^4} \right),$$

which implies

$$\frac{F(\epsilon)}{\epsilon} \sim \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \left(-10 \frac{10 a_\epsilon}{\epsilon^3} - \frac{a_\epsilon}{2\epsilon^3} - \frac{a_\epsilon}{2\epsilon^3} + \frac{a_\epsilon}{2\epsilon^3} + 0 \right),$$

and hence

$$\lim_{\epsilon \rightarrow 0} \frac{F(\epsilon)}{\epsilon} = -3 m \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2}. \quad (1.7.14)$$

Similarly,

$$\frac{G(\epsilon)}{\epsilon} = \frac{\mu_2}{\epsilon} g'(\epsilon) = \mu_2 \left(-4K_1 + \frac{K_3}{2\epsilon^3} - \frac{3K_4}{2\epsilon^5} \right),$$

which implies

$$\frac{G(\epsilon)}{\epsilon} \sim \mu_2 \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \left(\frac{a_\epsilon}{\epsilon^3} - \frac{a_\epsilon}{4\epsilon^3} \right),$$

and hence

$$\lim_{\epsilon \rightarrow 0} \frac{G(\epsilon)}{\epsilon} = m\mu_2 \frac{3(3\mu_1 + 2\mu_2)}{4(\mu_1 + \mu_2)}. \quad (1.7.15)$$

Relations (1.7.10) – (1.7.15) yield:

$$\langle \operatorname{div} (\mu \nabla \mathbf{w}^{k\epsilon} - q^{k\epsilon} I) , \boldsymbol{\psi} \rangle \rightarrow -m\mu_2 \frac{\pi}{4} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \int_{\Omega} \mathbf{e}_k \cdot \phi \mathbf{u} \, d\mathbf{x}, \quad (1.7.16)$$

for any $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $H_0^1(\Omega)^3$ with $\mathbf{u}^\epsilon \cdot \mathbf{n} = 0$ on S_ℓ^ϵ , and for any $\phi \in C_0^\infty(\Omega)$. □

Using LEMMA 1.7.1 we can now compute (1.6.3). Indeed, we have:

$$\begin{aligned} \frac{a_\epsilon}{\epsilon^3} \int_{B_\ell^{\frac{\epsilon}{a_\epsilon}}} \mu e(\boldsymbol{\chi}^{m\frac{\epsilon}{a_\epsilon}}) : e(\boldsymbol{\chi}^{k\frac{\epsilon}{a_\epsilon}}) \, d\mathbf{x} &= \frac{1}{\epsilon^3} \int_{B_\ell^\epsilon} \mu e(\mathbf{w}^{m\epsilon}) : e(\mathbf{w}^{k\epsilon}) \, d\mathbf{x} \\ &= \frac{1}{2|\Omega|} \int_{B^\epsilon} 2\mu e(\mathbf{w}^{m\epsilon}) : e(\mathbf{w}^{k\epsilon}) \, d\mathbf{x} = \frac{1}{2|\Omega|} \int_{B^\epsilon} \sigma_{ij}^{m\epsilon} \frac{\partial w_i^{k\epsilon}}{\partial x_j} \, d\mathbf{x} = \frac{1}{2|\Omega|} \langle -\operatorname{div} \sigma^{m\epsilon} , \mathbf{w}^{k\epsilon} \rangle, \end{aligned} \quad (1.7.17)$$

where the last term was obtained using integration by parts and properties of the local problem (1.7.2). As $\epsilon \rightarrow 0$ we obtain:

$$\mathcal{M}_{mk} = m\mu_2 \frac{\pi}{8} \frac{3\mu_1 + 2\mu_2}{\mu_1 + \mu_2} \delta_{mk}.$$

1.8 Time dependent case

The results obtained for the time stationary Stokes equation can be extended to the time dependent case by using the fact that Γ -convergence of E^ϵ to E in the weak topology $H_0^1(\Omega)^3$ is equivalent to Mosco convergence of E^ϵ to E in $L^2(\Omega)^3$ (see THEOREM A.5.6 in appendix A.5). Then using the connection between Mosco convergence and the convergence of solutions for a class of evolution problems we can get convergence in the time dependent case. We begin with some auxiliary results.

Let $\mathbf{H} = L^2(\Omega)^3$ and for any $\epsilon > 0$ let E^ϵ be the convex functional on \mathbf{H} , with $\operatorname{dom}(E^\epsilon) = V^\epsilon$, defined by

$$E^\epsilon(\mathbf{u}) = \begin{cases} \int_{\Omega} \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) \, dx & \mathbf{u} \in V^\epsilon, \\ +\infty & \mathbf{u} \notin V^\epsilon. \end{cases}$$

The sub-differential of E^ϵ is

$$\partial E^\epsilon(\mathbf{u}) = \{\boldsymbol{\xi} \in \mathbf{H} : (\boldsymbol{\xi}, \mathbf{v} - \mathbf{u}) \leq E^\epsilon(\mathbf{v}) - E^\epsilon(\mathbf{u}) \text{ for all } \mathbf{v} \in \text{dom}(E^\epsilon)\}.$$

Thus, if $\mathbf{u}^\epsilon \in V^\epsilon$ then $\boldsymbol{\xi} \in \partial E^\epsilon(\mathbf{u}^\epsilon)$ if and only if for every $\mathbf{v}^\epsilon \in \text{dom}(E^\epsilon)$

$$E^\epsilon(\mathbf{v}^\epsilon) \geq E^\epsilon(\mathbf{u}^\epsilon) + (\boldsymbol{\xi}, \mathbf{v}^\epsilon - \mathbf{u}^\epsilon). \quad (1.8.1)$$

Select $\mathbf{v}^\epsilon = \mathbf{u}^\epsilon + \lambda \boldsymbol{\phi}$ where $\boldsymbol{\phi} \in C_0^\infty(\Omega_{1\epsilon})^3$ with $\text{div} \boldsymbol{\phi} = 0$, and $\lambda \in \mathbb{R}$. Substitute \mathbf{v}^ϵ in (1.8.1) to obtain

$$\int_{\Omega_{1\epsilon}} 2 \mu^\epsilon e(\mathbf{u}^\epsilon) : e(\boldsymbol{\phi}) \, d\mathbf{x} = (\boldsymbol{\xi}, \boldsymbol{\phi}). \quad (1.8.2)$$

This implies that there exists a distribution $p_1^\epsilon \in L^2(\Omega_{1\epsilon})^3$ such that $-\text{div} \sigma^{1\epsilon} = \boldsymbol{\xi}$ in the sense of distribution in $\Omega_{1\epsilon}$. In a similar manner if we select $\boldsymbol{\phi} \in C_0^\infty(\Omega_{2\epsilon})^3$ with $\text{div} \boldsymbol{\phi} = 0$ we obtain that $-\text{div} \sigma^{2\epsilon} = \boldsymbol{\xi}$ in the sense of distribution in $\Omega_{2\epsilon}$ or we can write $-\text{div} \sigma^\epsilon = \boldsymbol{\xi}$ in the sense of distribution in $\Omega_{1\epsilon} \cup \Omega_{2\epsilon}$. Furthermore, in exactly the same way as in appendix A.1, we obtain the remaining boundary conditions, and balance of torque on each fluid particle.

1.8.1 Emulsion flow problem

If $I = (0, T)$, then mathematical formulation of the time-dependent emulsion flow problem is,

$$\begin{aligned} \frac{\partial \mathbf{u}^\epsilon(t, \mathbf{x})}{\partial t} - \text{div} \sigma^\epsilon &= \mathbf{f}(t, \mathbf{x}) && \text{in } I \times (\Omega_{1\epsilon} \cup \Omega_{2\epsilon}), \\ \text{div} \mathbf{u}^\epsilon(t, \mathbf{x}) &= 0 && \text{in } I \times \Omega, \\ \llbracket \mathbf{u}^\epsilon(t, \mathbf{x}) \rrbracket &= \mathbf{0} && \text{on } I \times S_\ell^\epsilon, \\ \mathbf{u}^\epsilon(t, \mathbf{x}) &= \mathbf{c} \times (\mathbf{x} - \mathbf{x}_C^\ell) && \text{on } I \times S_\ell^\epsilon, \\ \mathbf{u}^\epsilon(t, \mathbf{x}) &= \mathbf{0} && \text{on } I \times \partial\Omega, \\ \mathbf{u}^\epsilon(t, \mathbf{x}) &= \mathbf{u}^\epsilon(\mathbf{x}) && \text{on } \{0\} \times (\Omega_{1\epsilon} \cup \Omega_{2\epsilon}), \\ \int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times \llbracket \sigma^\epsilon \mathbf{n} \rrbracket \, ds &= 0. \end{aligned} \quad (1.8.3)$$

We associate with $\mathbf{u}^\epsilon(t, \mathbf{x})$ a mapping $\hat{\mathbf{u}}^\epsilon : [0, T] \mapsto \mathbf{H}$ defined by $[\hat{\mathbf{u}}^\epsilon(t)](\mathbf{x}) := \mathbf{u}^\epsilon(t, \mathbf{x})$ where $\mathbf{x} \in \Omega_{1\epsilon} \cup \Omega_{2\epsilon}$, and $0 \leq t \leq T$. In other words we are going to consider $\mathbf{u}^\epsilon(t, \mathbf{x})$ not as

a function of \mathbf{x} and t but as a mapping $\hat{\mathbf{u}}^\epsilon$ of t into \mathbf{H} of functions of \mathbf{x} . We interpret $\mathbf{f}(t, \mathbf{x})$ in a similar manner.

The variational formulation of (1.8.3) is

$$\left\{ \begin{array}{l} \text{Find } \hat{\mathbf{u}}^\epsilon(t) \in L^2(0, T; \mathbf{H}) \text{ such that} \\ \int_{\Omega} \frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} \cdot \boldsymbol{\phi}(\mathbf{x}) d\mathbf{x} + \int_{\Omega} 2\mu^\epsilon e(\hat{\mathbf{u}}^\epsilon(t)) : e(\boldsymbol{\phi}(\mathbf{x})) d\mathbf{x} \\ = \int_{\Omega} \hat{\mathbf{f}}(t) \cdot \boldsymbol{\phi}(\mathbf{x}) d\mathbf{x} \text{ for all } \boldsymbol{\phi} \in V^\epsilon. \end{array} \right. \quad (1.8.4)$$

Equivalently, using (1.8.2), we can write (1.8.4) as

$$\left\{ \begin{array}{l} \frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} + \partial E^\epsilon(\hat{\mathbf{u}}^\epsilon(t)) \ni \hat{\mathbf{f}}(t), \quad 0 < t < T \\ \hat{\mathbf{u}}^\epsilon(0) = \mathbf{u}^\epsilon. \end{array} \right. \quad (1.8.5)$$

Therefore we have obtained an equivalence between (1.8.3) and (1.8.5). In similar manner, if we define

$$E(\mathbf{u}) = \left\{ \begin{array}{ll} \int_{\Omega} \mu_2 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + \int_{\Omega} \mathbf{u}^\top \mathcal{M} \mathbf{u} d\mathbf{x} & \text{for } \mathbf{u} \in V, \\ + \infty & \text{for } \mathbf{u} \notin V. \end{array} \right.$$

in \mathbf{H} with $\text{dom}(E) = V$ and \mathcal{M} the matrix defined in equation (2.5.2), we obtain an equivalence between the time dependent homogenized suspensions problem,

$$\left\{ \begin{array}{ll} \frac{\partial \mathbf{u}(t, \mathbf{x})}{\partial t} - \mu_2 \Delta \mathbf{u} + \nabla p + \mathcal{M} \mathbf{u} = \mathbf{f}(t, \mathbf{x}) & \text{in } I \times \Omega, \\ \text{div} \mathbf{u}(t, \mathbf{x}) = 0 & \text{in } I \times \Omega, \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{0} & \text{on } I \times \partial\Omega, \\ \mathbf{u}(t, \mathbf{x}) = \mathbf{u}(\mathbf{x}) & \text{on } \{0\} \times \Omega, \end{array} \right. \quad (1.8.6)$$

and

$$\left\{ \begin{array}{l} \frac{d\hat{\mathbf{u}}(t)}{dt} + \partial E(\hat{\mathbf{u}}(t)) \ni \hat{\mathbf{f}}(t), \quad 0 < t < T \\ \hat{\mathbf{u}}(0) = \mathbf{u}. \end{array} \right. \quad (1.8.7)$$

Using well know results from the theory of nonlinear semigroups and maximal monotone operators in Hilbert spaces (see [9], [56]), well posedness as well as existence and uniqueness of solutions of both (1.8.5) and (1.8.7) are established.

We are now ready to state the main result of this section:

THEOREM 1.8.1. Assume that $\mathbf{u}^\epsilon \rightarrow \mathbf{u}$ in \mathbf{H} . Then the solution $\hat{\mathbf{u}}^\epsilon$ converges uniformly to $\hat{\mathbf{u}}$ on $[0, T]$ and

$$\int_0^T t \left\| \frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} - \frac{d\hat{\mathbf{u}}(t)}{dt} \right\|_{\mathbf{H}}^2 dt \rightarrow 0.$$

If in addition $\mathbf{u}^\epsilon \in \text{dom}(E^\epsilon)$, $\mathbf{u} \in \text{dom}(E)$ and $E^\epsilon(\mathbf{u}^\epsilon) \rightarrow E(\mathbf{u})$ then

$$\frac{d\hat{\mathbf{u}}^\epsilon(t)}{dt} \rightarrow \frac{d\hat{\mathbf{u}}(t)}{dt} \text{ strongly in } L^2(0, T; \mathbf{H}),$$

and

$$E^\epsilon(\hat{\mathbf{u}}^\epsilon(t)) \rightarrow E(\hat{\mathbf{u}}(t)) \text{ uniformly on } [0, T].$$

Proof. Straight forward application of THEOREM 3.66 and THEOREM 3.74 in [3] and the Mosco convergence of E^ϵ to E \square

1.9 Conclusions

The problem of dilute emulsions formed by two newtonian fluids, in which one fluid is dispersed under the form of droplets of arbitrary shape, in the presence of surface tension, is formulated in the homogenization framework.

We prove using Γ -convergence in THEOREM 1.6.3, in the case of droplets with fixed centers of mass, if $a_\epsilon = \mathcal{O}(\epsilon^3)$ the limit behavior is described by a Brinkman type law; while for the case of convected droplets the limit is given by the unperturbed Stokes flow. For $a_\epsilon = o(\epsilon^3)$, in both cases the limit is given by the unperturbed flow. The Brinkman law obtained here takes into account the geometry of the droplets and their viscosity; for $\mu_1 \rightarrow \infty$ one can show that the results reduce to the Brinkman law for rigid particles previously obtained in [1] and [11].

For spherical droplets we can actually compute the solutions of the local problems and thus the tensor appearing in the Brinkman law in LEMMA 1.7.1. This also gives the form for the suspension of rigid particles, in the limit for $\mu_1 \rightarrow \infty$.

The time-dependent problem is studied using the consequences of the Mosco-convergence for the energy functionals corresponding to the elliptic case.

Chapter 2

Asymptotics for a multi-scale model of dilute emulsions with surface effects: the case of convected droplets

2.1 Viscous drops in a viscous fluid with surface tension

We consider an emulsion of two immiscible fluids, one with viscosity μ_2 , forming a connected phase, in which droplets with arbitrary shape of the second fluid of viscosity μ_1 are distributed. We assume the flow to be quasi-static and at low Reynolds numbers. Thus, the system is modeled by a stationary Stokes flow in an open, bounded, Lipschitz domain Ω in \mathbb{R}^n ($n = 2, 3$). The droplets are periodically distributed with characteristic size a_ϵ and with the distance between the droplets' centers of size ϵ , and $\epsilon \gg a_\epsilon$. Moreover, we assume that the droplets do not intersect the boundary, $\partial\Omega$. Additionally, we consider the presence on the droplets' surface of a non-uniform surface tension, as in the case when surfactants are present.

The homogenization emulsion problem for the non-dilute case has been studied using two-scale convergence by Lipton and Vernescu [33].

Problems in which the effect of surfactants are important can also involve surfactant transport along the interface. In such models the interface possesses its own rheological properties [41]. In this work we do not consider such a scenario.

Our goal is to derive the asymptotic behavior of the velocity of the emulsion, \mathbf{v}^ϵ as ϵ tends to zero. Problems of this kind have been the subject of active research for quite some time starting with the work of Taylor [57]. More recently, work on this topic has been done by Ammari, Garapon, Kang and Lee [2], Bonnetier, Manceau and Triki [6], Nika and Vernescu [43], [42] and many others. Our work is an extension of Nika and Vernescu [42] and Bonnetier, Manceau and Triki [6]. The authors of [6] considered droplets with constant surface tension in an ambient fluid and they established an asymptotic form for the velocity of the emulsion in terms of viscous moment tensor and a curvature moment tensor. However,

their equations did not include a kinematic boundary condition on the droplet surface. In [23], as well as in [42], the authors imposed a kinematic condition on the droplet surface in order to ensure that the interface remains a material inter-phase boundary. In [42], the kinematic condition is necessary for recovering G.I. Taylor's viscosity of an emulsion [57].

The goal of this chapter is to study the asymptotics of the velocity in the case of dilute emulsions of arbitrary shaped droplets of size a_ϵ , in the presence of surface tension.

We present a general formulation of the problem and explain the choice of boundary conditions on the surface of the droplets. We derive the convergence of the emulsion velocity \mathbf{v}^ϵ to the velocity field \mathbf{v} corresponding to the unperturbed flow; this is done using Γ -convergence of the corresponding functionals and moreover, the rate of convergence is shown to be

$$\|\mathbf{v}^\epsilon - \mathbf{v}\|_{H_0^1(\Omega)^3} \leq C a_\epsilon^{3/2}.$$

Furthermore, we find the a_ϵ^3 term in the expansion of the velocity, which involves bulk and surface polarization tensors. The formula for the effective viscosity of the dilute emulsion is derived and the first term in the volume fraction expansion recovers G. I. Taylor's result [57].

The results in this chapter are based on Nika and Vernescu [42] and [44].

2.2 Periodic homogenization for droplets of arbitrary shape with surface tension

For the homogenization setting of the emulsion problem, as in chapter one, we define $\Omega \subset \mathbb{R}^3$, to be a bounded open set with smooth enough boundary $\partial\Omega$, and let $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^3$ be the unit cube in \mathbb{R}^3 . For every $\epsilon > 0$, let N^ϵ be the set of all points $\ell \in \mathbb{Z}^3$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with Lipschitz boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subset \subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon\ell + a_\epsilon T$. The set T_ℓ^ϵ represents one of the droplets suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface. We now define the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}},$$

where $\Omega_{1\epsilon}$ is the domain occupied by the droplets of viscosity μ_1 , and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid, of viscosity μ_2 . Let \mathbf{n} be the unit normal on the boundary of $\Omega_{1\epsilon}$ that points outside the domain. In this setting the emulsion problem is described by

$$-\operatorname{div}(2\mu^\epsilon e(\mathbf{v}^\epsilon) - p^\epsilon I) = \mathbf{f} + (\lambda_\epsilon \kappa_\epsilon \mathbf{n} - \nabla_s \lambda_\epsilon) \chi_{S^\epsilon} \quad \text{in } \Omega, \quad (2.2.1a)$$

$$\operatorname{div} \mathbf{v}^\epsilon = 0 \quad \text{in } \Omega, \quad (2.2.1b)$$

$$[[\mathbf{v}^\epsilon]] = 0 \quad \text{on } S_\ell^\epsilon, \quad (2.2.1c)$$

$$\mathbf{v}^\epsilon = \mathbf{V}^{\ell,\epsilon} + \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_c^\ell) \quad \text{on } S_\ell^\epsilon, \quad (2.2.1d)$$

$$\mathbf{v}^\epsilon = \mathbf{0} \quad \text{on } \partial\Omega, \quad (2.2.1e)$$

where \mathbf{v}^ϵ represents the velocity field, p^ϵ the pressure, $e(\mathbf{v}^\epsilon)$ the strain rate, \mathbf{f} the body forces, λ_ϵ the surface tension, κ_ϵ the mean curvature, \mathbf{n} the exterior normal to the droplets, $\mu^\epsilon(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_{1\epsilon}$ and $\mu^\epsilon(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_{2\epsilon}$. Moreover, for simplicity, we will assume that \mathbf{f} is $C^\infty(\Omega)^3$.

We note here that the second term on the right hand side of (3.1.1a) is due to the presence of surface tension as the jump of the normal stress vector on S_ℓ^ϵ is of the form (see APPENDIX A.4)

$$[[\sigma^\epsilon \mathbf{n}]] = \lambda_\epsilon \kappa_\epsilon \mathbf{n} - \nabla_s \lambda_\epsilon, \quad (2.2.2)$$

where the stress is $\sigma^\epsilon = -p^\epsilon I + 2\mu^\epsilon(\mathbf{x}) e(\mathbf{v}^\epsilon)$. Condition (2.2.1d) represents the kinematic condition with $\mathbf{V}^{\ell,\epsilon}$ and $\boldsymbol{\omega}^{\ell,\epsilon}$ representing respectively the unknown droplet surface translational and angular velocities, for droplet ℓ with \mathbf{x}_c^ℓ the position vector of its center of mass. Moreover, we have that the droplet velocities are related to the flow by the following:

$$\mathbf{V}^{\ell,\epsilon} = \frac{1}{|T_\ell^\epsilon|} \int_{T_\ell^\epsilon} \mathbf{v}^\epsilon d\mathbf{x}$$

2.2.1 Balance of forces and torques

In system (2.2.1) we have imposed the jump of the stress vector on the surface of the droplets. Let us verify that under this assumption the droplets are in equilibrium. To that end we recall a classical result from calculus on surfaces (see [19]).

PROPOSITION 2.2.1. *Assume that T has a $W^{2,\infty}$ boundary and that $g \in W^{2,1}(T)$ and $\mathbf{v} \in (C^1(\mathbb{R}^n))^n$ uniformly bounded then we have the following integration by parts on the surface $S = \partial T$:*

$$\int_S (\mathbf{v} \cdot \nabla g + g \operatorname{div}_s \mathbf{v}) ds = \int_S \left(\frac{\partial g}{\partial n} + \kappa g \right) \mathbf{v} \cdot \mathbf{n} ds, \quad (2.2.3)$$

where $\operatorname{div}_s \mathbf{v} = \operatorname{div} \mathbf{v} - \nabla \mathbf{v} \mathbf{n} \cdot \mathbf{n}$.

In particular, a manipulation of the expression in (2.2.3) yields

$$\int_S (g \kappa \mathbf{n} - \nabla_s g) \cdot \mathbf{v} ds = \int_S g \operatorname{div}_s \mathbf{v} ds. \quad (2.2.4)$$

Let us now apply (2.2.4) with $g = \lambda$ and $\mathbf{v} = \mathbf{e}_i$ we obtain that

$$\int_S [[\sigma \mathbf{n}]] ds = \int_S (\lambda \kappa \mathbf{n} - \nabla_s \lambda) \cdot \mathbf{e}_i ds = 0,$$

which represents the balance of forces on each droplet.

Similarly by denoting $\mathbf{y} = \mathbf{x} - \mathbf{x}_c$ from (2.2.4) for $g = \lambda$ and $\mathbf{v} = \mathbf{e}_i \times \mathbf{y}$ we have

$$\int_S (\mathbf{x} - \mathbf{x}_c) \times \llbracket \sigma \mathbf{n} \rrbracket \cdot \mathbf{e}_i ds = \int_S (\lambda \kappa \mathbf{n} - \nabla_s \lambda) \cdot (\mathbf{e}_i \times \mathbf{y}) ds = \int_S \lambda \operatorname{div}_s (\mathbf{e}_i \times \mathbf{y}) ds = 0,$$

which verifies the balance of torques on each droplet.

2.3 Weak formulation

The emulsion flow problem in (2.2.1) has the equivalent variational formulation: For any $\mathbf{f} \in L^2(\Omega)^3$, find $\mathbf{v}^\epsilon \in V^\epsilon$ such that,

$$\int_\Omega 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{w} d\mathbf{x}, \text{ for any } \mathbf{w} \in V^\epsilon, \quad (2.3.1)$$

where V^ϵ is the closed subspace of $H_0^1(\Omega)^3$ given by:

$$V^\epsilon = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \mathbf{w} = \mathbf{V}^{\ell, \epsilon} + \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_c^\ell) \text{ on } S_\ell^\epsilon, \boldsymbol{\omega}^{\ell, \epsilon} \in \mathbb{R}^3\}.$$

The proof of the equivalence is derived in the appendix and relies on the fact that, for non-uniform surface tension, the jump of the stress is of the form (2.2.2) and and by PROPOSITION 2.2.1 the balance of forces and torques are automatically satisfied.

The existence and uniqueness of a weak solution of the emulsion flow problem follows from the Lax-Milgram lemma. Furthermore, \mathbf{v}^ϵ is the unique minimizer of the problem:

$$\begin{cases} \text{Find } \mathbf{v}^\epsilon \in H_0^1(\Omega)^3 \text{ such that} \\ J^\epsilon(\mathbf{v}^\epsilon) = \min_{\mathbf{u} \in H_0^1(\Omega)^3} J^\epsilon(\mathbf{u}) \end{cases} \quad (2.3.2)$$

where

$$J^\epsilon(\mathbf{u}) = \int_\Omega \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} - \int_\Omega \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}) \quad (2.3.3)$$

and I_D represents the indicator function of the set D , defined by

$$I_D(s) = \begin{cases} 0 & \text{if } s \in D \\ +\infty & \text{if } s \notin D \end{cases}$$

2.4 The local problem

Let us consider for some $\ell \in N^\epsilon$ the reference cell, $Y_\ell^\epsilon = \epsilon(\ell + Y)$ and the ball B_ℓ^ϵ with center in the center of cell $\ell \in N^\epsilon$ and radius $\epsilon/2$. The local problem is defined as finding

$(\mathbf{w}^{k\epsilon}, q^{k\epsilon})$ solutions of

$$\left\{ \begin{array}{ll} -\operatorname{div} \sigma^{k\epsilon} = (\lambda_\epsilon \kappa_\epsilon \mathbf{n} - \nabla_s \lambda_\epsilon) \chi_{S^\epsilon} & \text{in } B_\ell^\epsilon \setminus S_\ell^\epsilon, \\ \sigma^{k\epsilon} = -q^{k\epsilon} I + 2\mu e(\mathbf{w}^{k\epsilon}), & \\ \operatorname{div} \mathbf{w}^{k\epsilon} = 0 & \text{in } B_\ell^\epsilon, \\ \llbracket \mathbf{w}^{k\epsilon} \rrbracket = \mathbf{0} & \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} = \mathbf{W}^{\ell, \epsilon} + \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{y} - \mathbf{y}_c^\ell) & \text{on } S_\ell^\epsilon, \\ \mathbf{w}^{k\epsilon} = \mathbf{e}_k & \text{on } \partial B_\ell^\epsilon, \end{array} \right. \quad (2.4.1)$$

where \mathbf{e}_k is the k^{th} - unit vector of the Cartesian base and $\mu = \mu_d$ in T_ℓ^ϵ and $\mu = \mu_0$ in $B_\ell^\epsilon - \overline{T_\ell^\epsilon}$. Existence and uniqueness of a solution to (2.4.1) is established as in [43] and furthermore the following result is also proved:

LEMMA 2.4.1. *If $a_\epsilon = \mathcal{O}(\epsilon^3)$ then the solution to the local problem (2.4.1), $\mathbf{w}^{k\epsilon}$ converges strongly to \mathbf{e}_k in $H^1(\Omega)^3$.*

2.5 Convergence of the energies

Using (A.2.5), let us define the energy functional $E^\epsilon : H_0^1(\Omega)^3 \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$E^\epsilon(\mathbf{u}) = \int_\Omega \mu^\epsilon e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + I_{V^\epsilon}(\mathbf{u}). \quad (2.5.1)$$

THEOREM 2.5.1. *The sequence of functionals $(E^\epsilon)_\epsilon$ Γ -converges in the strong topology of $H_0^1(\Omega)^3$ to the functional $E : H_0^1(\Omega)^3 \mapsto \mathbb{R} \cup \{+\infty\}$ defined by*

$$E(\mathbf{u}) = \int_\Omega \mu_0 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} + I_V(\mathbf{u}), \quad (2.5.2)$$

where V is the closed subspace of $H_0^1(\Omega)^3$ defined by

$$V = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0\}.$$

Proof. As in chapter one we only deal with divergence-free functions and prove the following assertions,

- (a) For all $\mathbf{v}^0 \in V$ there exists a $\mathbf{v}^\epsilon \in V^\epsilon$, $\mathbf{v}^\epsilon \rightarrow \mathbf{v}^0$ in $H_0^1(\Omega)^3$ such that $\lim_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{v}^\epsilon) = E(\mathbf{v}^0)$,
- (b) For all $\mathbf{u}^0 \in V$, for all $\mathbf{u}^\epsilon \in V^\epsilon$, $\mathbf{u}^\epsilon \rightarrow \mathbf{u}^0$ in $H_0^1(\Omega)^3$ such that $\liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u}^0)$.

Part (a). Let $\mathbf{v}^0 \in C_0^\infty(\Omega)^3$ such that $\operatorname{div} \mathbf{v}^0 = 0$. Define the sequence \mathbf{v}^ϵ as in (1.6.4),

$$\mathbf{v}^\epsilon(\mathbf{x}) = \begin{cases} \mathbf{v}^0(\mathbf{x}) & \text{in } Y_\ell^\epsilon - B_\ell^\epsilon, \\ \mathbf{v}^0(\mathbf{x}) + (\mathbf{w}^{k^\epsilon}(\mathbf{x}) - \mathbf{e}_k)v_k^0(\mathbf{x}_c^\ell) - \operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell} & \text{in } B_\ell^\epsilon - T_\ell^\epsilon, \\ v_k^0(\mathbf{x}_c^\ell)\mathbf{w}^{k^\epsilon}(\mathbf{x}) & \text{in } T_\ell^\epsilon, \end{cases}$$

One can now verify, that the sequence \mathbf{v}^ϵ belongs in V^ϵ and $\mathbf{v}^\epsilon \rightarrow \mathbf{v}^0$ in $H_0^1(\Omega)^3$. Hence,

$$\begin{aligned} E^\epsilon(\mathbf{v}^\epsilon) &= \sum_{\ell \in N^\epsilon} \int_{Y_\ell^\epsilon - T_\ell^\epsilon} \mu e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon} \mu v_k^0(\mathbf{x}_c^\ell) v_m^0(\mathbf{x}_c^\ell) e(\mathbf{w}^{k^\epsilon}) : e(\mathbf{w}^{m^\epsilon}) d\mathbf{x} \\ &+ \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2\mu v_k^0(\mathbf{x}_c^\ell) e(\mathbf{v}^0) : e(\mathbf{w}^{k^\epsilon}) d\mathbf{x} - \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2\mu e(\mathbf{v}^0) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell}) d\mathbf{x} \\ &- \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} 2\mu v_k^0(\mathbf{x}_c^\ell) e(\mathbf{w}^{k^\epsilon}) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell}) d\mathbf{x} \\ &+ \sum_{\ell \in N^\epsilon} \int_{B_\ell^\epsilon - T_\ell^\epsilon} \mu e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell}) : e(\operatorname{curl} \tilde{\mathbf{v}}_{\epsilon\ell}\phi_{\epsilon\ell}) d\mathbf{x} \\ &= \int_{\Omega \setminus \Omega_{1\epsilon}} \mu e(\mathbf{v}^0) : e(\mathbf{v}^0) d\mathbf{x} + o(1) \end{aligned}$$

Taking the limit as $\epsilon \rightarrow 0$

$$\lim_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{v}^\epsilon) = E(\mathbf{v}^0).$$

To pass to the limit for an arbitrary $\mathbf{v}^0 \in H_0^1(\Omega)^3$ we use a classic diagonalization argument.

Part (b). Let $\mathbf{u}^\epsilon \in V^\epsilon$ such that $\mathbf{u}^\epsilon \rightarrow \mathbf{u}^0$ in $H_0^1(\Omega)^3$, where $\mathbf{u}^0 \in V$. Moreover, assume that $\mathbf{v}^0 \in C_0^\infty(\Omega)^3$ such that $\operatorname{div} \mathbf{v}^0 = 0$ and define the sequence \mathbf{v}^ϵ as in (1.6.4). Using a sub-differential type inequality we get:

$$E^\epsilon(\mathbf{u}^\epsilon) \geq E^\epsilon(\mathbf{v}^\epsilon) + \int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{u}^\epsilon - \mathbf{v}^\epsilon) d\mathbf{x}. \quad (2.5.3)$$

Due to the strong convergence of $\mathbf{u}^\epsilon \rightarrow \mathbf{u}^0$, and $\mathbf{v}^\epsilon \rightarrow \mathbf{v}^0$ in $H_0^1(\Omega)^3$ we can pass to the limit first as $\epsilon \rightarrow 0$ then using the continuity of the functional E^ϵ and a diagonalization argument to make $\mathbf{v}^0 \rightarrow \mathbf{u}^0$ in the strong topology of $H_0^1(\Omega)^3$ we get,

$$\liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u}^0)$$

□

COROLLARY 2.5.2. *The sequence $\{\mathbf{v}^\epsilon\}$ of solutions to (2.3.2) is convergent in the strong topology of $H_0^1(\Omega)^3$ to \mathbf{v} solution of*

$$\begin{cases} \text{Find } \mathbf{v} \in H_0^1(\Omega)^3 \text{ such that,} \\ J(\mathbf{v}) = \min_{\mathbf{u} \in H_0^1(\Omega)^3} J(\mathbf{u}), \end{cases} \quad (2.5.4)$$

where

$$J(\mathbf{u}) = \int_{\Omega} \mu_0 e(\mathbf{u}) : e(\mathbf{u}) d\mathbf{x} - \int_{\Omega} \mathbf{f} \cdot \mathbf{u} d\mathbf{x} + I_V(\mathbf{u}), \quad (2.5.5)$$

and

$$V = \{\mathbf{w} \in H_0^1(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}.$$

REMARK: Computing the Euler-Lagrange equations of the functional J in (2.5.5) correspond to the unperturbed flow (i.e. $\mathcal{M} \equiv 0$). Hence, the limit problem is simply a Stokes problem

$$\begin{aligned} -\operatorname{div} (2\mu_2 e(\mathbf{v}) - pI) &= \mathbf{f} & \text{in } \Omega, \\ \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega, \\ \mathbf{v} &= \mathbf{0} & \text{on } \partial\Omega. \end{aligned} \quad (2.5.6)$$

2.6 Scaling of surface tension and mean curvature

On the droplet surface there is a stress jump $\llbracket \sigma^\epsilon \mathbf{n} \rrbracket \neq 0$. The stress jump can be obtained from the principle that the forces on an element of inter-facial area of arbitrary shape and size must be in equilibrium, because the interface is assumed to have zero thickness and thus zero mass. One can thus obtain (see APPENDIX A.4)

$$\llbracket \sigma^\epsilon \mathbf{n} \rrbracket = \lambda_\epsilon \kappa_\epsilon \mathbf{n} - \nabla_s \lambda_\epsilon,$$

where λ is the surface tension, and κ is the mean curvature. For a droplet of size a_ϵ we propose the following scaling of the surface tension, λ_ϵ , so that the surface energy remain bounded:

$$\lambda_\epsilon(\mathbf{x}) = \lambda_\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) = a_\epsilon \lambda(\mathbf{y}) \quad \text{for all } \mathbf{y} \in S,$$

where $\lambda(\mathbf{y})$ is the surface tension on S . This is in agreement with Tolman's scaling when the surface tension is uniform [59]. Moreover, the mean curvature on S^ϵ , κ_ϵ , scales the following way:

$$\kappa_\epsilon(\mathbf{x}) = \kappa_\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) = \frac{1}{a_\epsilon} \kappa(\mathbf{y}) \quad \text{for all } \mathbf{y} \in S,$$

where $\kappa(\mathbf{y})$ is the mean curvature on S . Thus, for a droplet of size a_ϵ the jump of the stress is the following:

$$\lambda_\epsilon(\mathbf{x}) \kappa_\epsilon(\mathbf{x}) \mathbf{n}_x - \nabla_{s,x} \lambda_\epsilon(\mathbf{x}) = \lambda(\mathbf{y}) \kappa(\mathbf{y}) \mathbf{n}_y - \nabla_{s,y} \lambda(\mathbf{y}).$$

2.7 H^1 rate of convergence for the velocity field

We are interested in deriving the order of convergence of the emulsion velocity \mathbf{v}^ϵ to the velocity field \mathbf{v} of the unperturbed flow. Since we are studying the dilute case it is sufficient to consider the problem for a single droplet. In doing so we will drop the subscripts ℓ from T_ℓ^ϵ and S_ℓ^ϵ and simply write T^ϵ and S^ϵ or T and S for their unscaled counterparts.

First, we will construct a test function \mathbf{w}^ϵ and then obtain some estimates for the difference of $\mathbf{v}^\epsilon - \mathbf{w}^\epsilon$ as well as $\mathbf{v} - \mathbf{w}^\epsilon$. For this we need the following results (see [11], [60]).

Using LEMMAS 1.6.1 and 1.6.2 we construct the vector field \mathbf{w}^ϵ ,

$$\mathbf{w}^\epsilon(\mathbf{x}) = \begin{cases} \mathbf{v}(\mathbf{x}) & \text{in } \Omega - B(2a_\epsilon), \\ \mathbf{v}(\mathbf{x}) - \text{curl}(\tilde{\mathbf{v}}\phi_\epsilon) & \text{in } B(2a_\epsilon) - T^\epsilon, \\ \mathbf{v}(\mathbf{x}_c) & \text{in } T^\epsilon. \end{cases} \quad (2.7.1)$$

REMARK: The construction of \mathbf{w}^ϵ makes it an appropriate test function for problem (2.2.1). Given the general nature of the stress for problem (2.2.1), \mathbf{w}^ϵ makes the variational formulation of (2.2.1) more accessible for constructing the estimates mentioned in the previous paragraph.

LEMMA 2.7.1. *There exists a constant $C > 0$, independent of ϵ , such that*

$$\|\mathbf{v}^\epsilon - \mathbf{w}^\epsilon\|_{H_0^1(\Omega)^3} \leq Ca_\epsilon^{3/2}.$$

Proof. Using $\mathbf{v}^\epsilon - \mathbf{w}^\epsilon$ as a test function in (2.2.1) we have

$$\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot (\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x},$$

where the surface term is zero due to the fact that $\mathbf{v}^\epsilon - \mathbf{w}^\epsilon$ is a rigid body motion on S_ℓ^ϵ and by using the balance of forces and torques condition. Moreover, we compute the following integral

$$\begin{aligned} & \int_{\Omega} 2\mu^\epsilon e(\mathbf{w}^\epsilon) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} \\ &= \int_{\Omega - B(2a_\epsilon)} 2\mu^\epsilon e(\mathbf{v}) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} + \int_{B(2a_\epsilon) - T^\epsilon} 2\mu^\epsilon e(\mathbf{v} - \text{curl} \tilde{\mathbf{v}}\phi_\epsilon) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} \\ &= \int_{\Omega} \mathbf{f} \cdot (\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} - \int_{B(2a_\epsilon) - T^\epsilon} 2\mu_2 e(\text{curl} \tilde{\mathbf{v}}\phi_\epsilon) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} \\ &\quad - \int_{T^\epsilon} 2\mu_2 e(\mathbf{v}) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x}. \end{aligned}$$

Hence, as the $e(\text{curl}(\tilde{\mathbf{v}}\phi_\epsilon)) = e(\mathbf{v})$ on T^ϵ , we obtain

$$\begin{aligned}
\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} &= \int_{B(2a_\epsilon)} 2\mu_0 e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon) : e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon) d\mathbf{x} \\
&\leq C \|e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon)\|_{L(B(2a_\epsilon))^3} \|e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon)\|_{L(B(2a_\epsilon))^3}.
\end{aligned}$$

Using results from elliptic regularity theory we have that the $\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon$ is bounded in Ω , therefore we obtain:

$$\|e(\mathbf{v}^\epsilon - \mathbf{w}^\epsilon)\|_{L^2(\Omega)^{3 \times 3}} \leq C a_\epsilon^{3/2},$$

and hence, using Korn's inequality we obtain the desired estimate. \square

LEMMA 2.7.2. *There exists a constant $C > 0$, independent of ϵ , such that*

$$\|\mathbf{v} - \mathbf{w}^\epsilon\|_{H_0^1(\Omega)^3} \leq C a_\epsilon^{3/2}.$$

Proof. We start by evaluating the following integral:

$$\begin{aligned}
&\int_{\Omega} e(\mathbf{v} - \mathbf{w}^\epsilon) : e(\mathbf{v} - \mathbf{w}^\epsilon) d\mathbf{x} \\
&= \int_{B(2a_\epsilon) - T^\epsilon} e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon) : e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon) d\mathbf{x} + \int_{T^\epsilon} e(\mathbf{v}) : e(\mathbf{v}) d\mathbf{x} \\
&= \int_{B(2a_\epsilon)} e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon) : e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon) d\mathbf{x} = \|e(\operatorname{curl} \tilde{\mathbf{v}} \phi_\epsilon)\|_{B(2a_\epsilon)^3}^2 \leq C a_\epsilon^3,
\end{aligned}$$

where the last inequality was obtained following the same arguments as in the previous lemma. Using Korn's inequality we obtain the desired estimate. \square

THEOREM 2.7.3. *There exists a constant $C > 0$, independent of ϵ , such that*

$$\|\mathbf{v}^\epsilon - \mathbf{v}\|_{H_0^1(\Omega)^3} \leq C a_\epsilon^{3/2}.$$

Proof. Follows from LEMMA 2.7.1, LEMMA 2.7.2, and an application of the triangle inequality. \square

2.8 Asymptotics of the velocity in the L^2 norm and correction terms

THEOREM 2.7.3, shows that the convergence of \mathbf{v}^ϵ to \mathbf{v} in the $H_0^1(\Omega)^3$ norm is of order $\mathcal{O}(a_\epsilon^{3/2})$. As a next step we are interested in computing the next term in the asymptotic expansion of the velocity vector field.

The approach we follow here is similar to the one in [6]. We denote by (\mathbf{G}, F) the Green's tensors associated with the homogenized flow in (2.5.6)

$$\begin{aligned} -\operatorname{div} (2 \mu_0 e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) - F_i(\mathbf{x}, \mathbf{z})I) &= \mathbf{e}_i \delta(\mathbf{x} - \mathbf{z}) && \text{in } \Omega, \\ \operatorname{div}_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) &= 0 && \text{in } \Omega, \\ \mathbf{G}_i(\mathbf{x}, \mathbf{z}) &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \quad (2.8.1)$$

Moreover, we introduce the following problems centered at the origin,

$$\begin{aligned} -\operatorname{div} (2 \mu e(\boldsymbol{\phi}^{kl}) - s^{kl}I) &= (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \chi_S && \text{in } \mathbb{R}^n, \\ \operatorname{div} \boldsymbol{\phi}^{kl} &= 0 && \text{in } \mathbb{R}^n, \\ \llbracket \boldsymbol{\phi}^{kl} \rrbracket &= 0 && \text{on } S, \\ \boldsymbol{\phi}^{kl} &= \boldsymbol{\phi}^{c,kl} + \boldsymbol{\omega}^{kl} \times \mathbf{y} && \text{on } S, \\ \boldsymbol{\phi}^{kl} &\rightarrow \mathbf{B}^{kl} && \text{at } \infty, \end{aligned} \quad (2.8.2)$$

$$\begin{aligned} -\operatorname{div}_y (2 \mu e_y(\mathbf{V}^\epsilon) - q^\epsilon I) &= (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \chi_S && \text{in } \Omega_\epsilon, \\ \operatorname{div}_y \mathbf{V}^\epsilon &= 0 && \text{in } \Omega_\epsilon, \\ \llbracket \mathbf{V}^\epsilon \rrbracket &= \mathbf{0} && \text{on } S, \\ \mathbf{V}^\epsilon &= \mathbf{V}^{c,\epsilon} + \boldsymbol{\omega} \times \mathbf{y} && \text{on } S, \\ \mathbf{V}^\epsilon &= e(\mathbf{v})(\mathbf{x}_c) \mathbf{y} && \text{on } \partial\Omega_\epsilon, \end{aligned} \quad (2.8.3)$$

$$\begin{aligned} -\operatorname{div}_y (2 \mu e_y(\mathbf{V}) - q I) &= (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \chi_S && \text{in } \mathbb{R}^n, \\ \operatorname{div}_y \mathbf{V} &= 0 && \text{in } \mathbb{R}^n, \\ \llbracket \mathbf{V} \rrbracket &= \mathbf{0} && \text{on } S, \\ \mathbf{V} &= \mathbf{V}^c + \boldsymbol{\omega} \times \mathbf{y} && \text{on } S, \\ \mathbf{V} &\rightarrow e(\mathbf{v})(\mathbf{x}_c) \mathbf{y} && \text{at } \infty, \end{aligned} \quad (2.8.4)$$

where $\mathbf{B}^{kl} = (\mathbf{e}_l y_k + \mathbf{e}_k y_l - \frac{2}{3} \mathbf{y} \delta_{kl})$, $\boldsymbol{\omega}^{kl}$, $\boldsymbol{\omega}$ are unknown constant vectors in \mathbb{R}^3 , $\boldsymbol{\phi}^{c,kl}$, $\mathbf{V}^{c,\epsilon}$ and \mathbf{V}^c are the translational velocities of the droplets (see [33]), and Ω_ϵ is the scaled domain defined the following way

$$\Omega_\epsilon := \left\{ \mathbf{y} = \frac{\mathbf{x} - \mathbf{x}_c}{a_\epsilon} \mid \mathbf{x} \in \Omega \right\}.$$

For simplicity we have taken Ω_ϵ to be a ball in \mathbb{R}^3 . We further remark that the balance of forces and torques for each problem is automatically satisfied by virtue of formula (2.2.4), whilst existence and uniqueness of the above problems follows from the theory of pseudo-monotone operators in Hilbert spaces (see [30]).

We re-define problems (2.8.2), (2.8.3), (2.8.4) by introducing the vector fields $\hat{\phi}^{kl} := \phi^{kl} - \mathbf{B}^{kl}$, $\mathbf{W}^\epsilon := \mathbf{V}^\epsilon - e(\mathbf{v})(\mathbf{x}_c)\mathbf{y}$, and $\mathbf{W} := \mathbf{V} - e(\mathbf{v})(\mathbf{x}_c)\mathbf{y}$ that decay to zero at infinity. Hence, we obtain the following corresponding problems for (2.8.2), (2.8.3), and (2.8.4),

$$\begin{aligned}
-\operatorname{div} \left(2 \mu e(\hat{\phi}^{kl}) - s^{kl} I \right) &= (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \chi_S \\
&\quad - (\mu_d - \mu_0) (\mathbf{e}_l n_k + \mathbf{e}_k n_l - \frac{2}{3} \mathbf{n}_y \delta_{kl}) \chi_S \quad \text{in } \mathbb{R}^3, \\
\operatorname{div} \hat{\phi}^{kl} &= 0 \quad \text{in } \mathbb{R}^3, \\
\llbracket \hat{\phi}^{kl} \rrbracket &= 0 \quad \text{on } S, \\
\hat{\phi}^{kl} &= -\mathbf{B}^{kl} + \phi^{c,kl} + \omega^{kl} \times \mathbf{y} \quad \text{on } S, \\
\hat{\phi}^{kl} &= \mathcal{O}(|\mathbf{y}|^{-1}) \quad \text{at } \infty, \\
s^{kl} &= \mathcal{O}(|\mathbf{y}|^{-2}) \quad \text{at } \infty,
\end{aligned} \tag{2.8.5}$$

$$\begin{aligned}
-\operatorname{div}_y (2 \mu e_y(\mathbf{W}^\epsilon) - q^\epsilon I) &= (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \chi_S \\
&\quad - 2 (\mu_1 - \mu_2) e(\mathbf{v})(\mathbf{x}_c) \mathbf{n}_y \chi_S \quad \text{in } \Omega_\epsilon, \\
\operatorname{div}_y \mathbf{W}^\epsilon &= 0 \quad \text{in } \Omega_\epsilon, \\
\llbracket \mathbf{W}^\epsilon \rrbracket &= \mathbf{0} \quad \text{on } S, \\
\mathbf{W}^\epsilon &= -e(\mathbf{v})(\mathbf{x}_c) \mathbf{y} + \mathbf{V}^{c,\epsilon} + \omega \times \mathbf{y} \quad \text{on } S, \\
\mathbf{W}^\epsilon &= \mathbf{0} \quad \text{on } \partial\Omega_\epsilon,
\end{aligned} \tag{2.8.6}$$

$$\begin{aligned}
-\operatorname{div}_y (2 \mu e_y(\mathbf{W}) - q I) &= (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \chi_S \\
&\quad - 2 (\mu_1 - \mu_2) e(\mathbf{v})(\mathbf{x}_c) \mathbf{n}_y \chi_S \quad \text{in } \mathbb{R}^3, \\
\operatorname{div}_y \mathbf{W} &= 0 \quad \text{in } \mathbb{R}^3, \\
\llbracket \mathbf{W} \rrbracket &= \mathbf{0} \quad \text{on } S, \\
\mathbf{W} &= -e(\mathbf{v})(\mathbf{x}_c) \mathbf{y} + \mathbf{V}^c + \omega \times \mathbf{y} \quad \text{on } S, \\
\mathbf{W} &= \mathcal{O}(|\mathbf{y}|^{-1}) \quad \text{at } \infty, \\
q &= \mathcal{O}(|\mathbf{y}|^{-2}) \quad \text{at } \infty,
\end{aligned} \tag{2.8.7}$$

LEMMA 2.8.1. *There exists a constant C , independent of ϵ , such that*

$$\|e_y(\mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) - \mathbf{v}(a_\epsilon \mathbf{y} + \mathbf{x}_c) - a_\epsilon \mathbf{W}^\epsilon(\mathbf{y}))\|_{L^2(\Omega_\epsilon)^{3 \times 3}} \leq C a_\epsilon^2.$$

Proof. Define $\Xi^\epsilon(\mathbf{y}) = \mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) - \mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) - a_\epsilon \mathbf{W}^\epsilon(\mathbf{y})$ and consider

$$\begin{aligned} \int_{\Omega_\epsilon} 2\mu |e_y(\Xi^\epsilon)|^2 d\mathbf{y} &= \int_{\Omega_\epsilon} 2\mu e_y(\mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c)) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y} \\ &\quad - \int_{\Omega_\epsilon} 2\mu e_y(\mathbf{v}(a_\epsilon \mathbf{y} + \mathbf{x}_c)) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y} \\ &\quad - a_\epsilon \int_{\Omega_\epsilon} 2\mu e_y(\mathbf{W}^\epsilon(\mathbf{y})) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y}. \end{aligned}$$

If $\mathbf{x} \in \Omega$, set $\xi^\epsilon(\mathbf{x}) := \Xi^\epsilon\left(\frac{\mathbf{x} - \mathbf{x}_c}{a_\epsilon}\right) = \Xi^\epsilon(\mathbf{y})$. We further note that Ξ^ϵ is divergence free and $\xi^\epsilon = \mathbf{0}$ on $\partial\Omega$. Thus, considering each integral separately we have,

$$\begin{aligned} &\int_{\Omega_\epsilon} 2\mu e_y(\mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c)) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y} \\ &= a_\epsilon^{-1} \int_{\Omega} 2\mu^\epsilon e_x(\mathbf{v}^\epsilon) : e_x(\xi^\epsilon) d\mathbf{x} \\ &= a_\epsilon^{-1} \int_{\Omega} \mathbf{f} \cdot \xi^\epsilon d\mathbf{x} + a_\epsilon^{-1} \int_{S^\epsilon} (\lambda_\epsilon \kappa_\epsilon \mathbf{n}_x - \nabla_{s,x} \lambda_\epsilon) \cdot \xi^\epsilon(\mathbf{x}) ds_x \\ &= a_\epsilon^{-1} \int_{\Omega} \mathbf{f} \cdot \xi^\epsilon d\mathbf{x} + a_\epsilon \int_S (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \cdot \Xi^\epsilon(\mathbf{y}) ds_y, \end{aligned}$$

$$\begin{aligned} &\int_{\Omega_\epsilon} 2\mu e_y(\mathbf{v}(a_\epsilon \mathbf{y} + \mathbf{x}_c)) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y} \\ &= a_\epsilon^{-1} \int_{\Omega} 2\mu^\epsilon e_x(\mathbf{v}) : e_x(\xi^\epsilon) d\mathbf{x} \\ &= a_\epsilon^{-1} \int_{\Omega} 2\mu_2 e_x(\mathbf{v}) : e_x(\xi^\epsilon) d\mathbf{x} + a_\epsilon^{-1} \int_{T^\epsilon} 2(\mu_1 - \mu_2) e_x(\mathbf{v}) : e_x(\xi^\epsilon) d\mathbf{x} \\ &= a_\epsilon^{-1} \int_{\Omega} \mathbf{f} \cdot \xi^\epsilon d\mathbf{x} + a_\epsilon \int_T 2(\mu_1 - \mu_2) e_x(\mathbf{v})(a_\epsilon \mathbf{y} + \mathbf{x}_c) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y}, \end{aligned}$$

$$\begin{aligned} &a_\epsilon \int_{\Omega_\epsilon} 2\mu e_y(\mathbf{W}^\epsilon(\mathbf{y})) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y} \\ &= a_\epsilon \int_S (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda - 2(\mu_2 - \mu_1) e(\mathbf{v})(\mathbf{x}_c) \mathbf{n}_y) \cdot \Xi^\epsilon(\mathbf{y}) ds_y. \end{aligned}$$

Therefore,

$$\int_{\Omega_\epsilon} 2\mu |e_y(\Xi^\epsilon)|^2 d\mathbf{y} = a_\epsilon \int_T 2(\mu_2 - \mu_1) (e_x(\mathbf{v})(a_\epsilon \mathbf{y} + \mathbf{x}_c) - e_x(\mathbf{v})(\mathbf{x}_c)) : e_y(\Xi^\epsilon(\mathbf{y})) d\mathbf{y}.$$

Using elliptic regularity results we can conclude that since \mathbf{v} is smooth inside Ω ,

$$\|e_y(\Xi^\epsilon)\|_{L^2(\Omega_\epsilon)^{3 \times 3}} \leq C a_\epsilon^2.$$

□

LEMMA 2.8.2. *There exists a constant C , independent of ϵ , such that*

$$\|e_y(\mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) - \mathbf{v}(a_\epsilon \mathbf{y} + \mathbf{x}_c) - a_\epsilon \mathbf{W}(\mathbf{y}))\|_{L^2(\Omega_\epsilon)^{3 \times 3}} \leq C a_\epsilon^{3/2}.$$

Proof. In view of LEMMA 2.8.1 it is enough to show that

$$\|e_y(\mathbf{W} - \mathbf{W}^\epsilon)\|_{L^2(\Omega_\epsilon)^{3 \times 3}} \leq C a_\epsilon^{1/2}.$$

This result follows as a special case from LEMMA 5.7 in [37]. The authors of [37] obtain a bound for the stress for a problem similar to (2.8.6). The concept they use is to transform (2.8.6) into a system in free space. Then by making use of the regularity lemma in [58] and the fundamental solution of the transformed Stokes system, which they obtain by way of the Fourier transform, are able to get an estimate for the stress in terms of the droplet radius. For a detailed proof see LEMMA A.3.1 and LEMMA A.3.2. □

THEOREM 2.8.3. *For any $\mathbf{z} \in \Omega$ at a distance $d > 0$ away from T^ϵ we have*

$$v_i^\epsilon(\mathbf{z}) = v_i(\mathbf{z}) + a_\epsilon^3 (e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \mathbb{P} e_x(\mathbf{v})(\mathbf{x}_c) - e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \mathbb{S}) + \mathcal{O}\left(a_\epsilon^{3+\frac{1}{2}}\right), \quad (2.8.8)$$

where \mathbb{P} and \mathbb{S} are the bulk and surface polarization tensors defined by:

$$\mathbb{P}_{ijkl} = 2(\mu_2 - \mu_1) \int_T e_{yij}(\phi^{kl})(\mathbf{y}) d\mathbf{y}, \quad \mathbb{S} = \int_S \lambda(\mathbf{y}) \mathbf{n}_y \otimes \mathbf{n}_y ds_y.$$

Proof. Using (2.8.1) and (2.5.6) we have

$$v_i(\mathbf{z}) = \int_\Omega 2\mu_2 e_x(\mathbf{G}_i) : e_x(\mathbf{v}) d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{G}_i d\mathbf{x}.$$

Similarly, using (2.8.1) and (3.1.1a) – (2.2.1e) we have

$$\begin{aligned} v_i^\epsilon(\mathbf{z}) &= \int_\Omega 2\mu_2 e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x(\mathbf{v}^\epsilon) d\mathbf{x} \\ &= \int_\Omega 2\mu^\epsilon e_x(\mathbf{v}^\epsilon) : e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) d\mathbf{x} \\ &\quad + \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{v}^\epsilon) : e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) d\mathbf{x} \\ &= \int_\Omega \mathbf{f} \cdot \mathbf{G}_i d\mathbf{x} + \int_{S^\epsilon} (\lambda_\epsilon \kappa_\epsilon \mathbf{n}_x - \nabla_{s,x} \lambda_\epsilon) \cdot \mathbf{G}_i(\mathbf{x}, \mathbf{z}) ds_x \\ &\quad + \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{v}^\epsilon) : e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) d\mathbf{x}. \end{aligned}$$

Thus we get,

$$\begin{aligned}
(v_i^\epsilon - v_i)(\mathbf{z}) &= \int_{S^\epsilon} (\lambda_\epsilon \kappa_\epsilon \mathbf{n}_x - \nabla_{s,x} \lambda_\epsilon) \cdot \mathbf{G}_i(\mathbf{x}, \mathbf{z}) d\mathbf{x} \\
&\quad + \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{v}^\epsilon) : e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) d\mathbf{x}.
\end{aligned} \tag{2.8.9}$$

We compute the integral over the surface first and we get,

$$\begin{aligned}
&\int_{S^\epsilon} (\lambda_\epsilon \kappa_\epsilon \mathbf{n}_x - \nabla_{s,x} \lambda_\epsilon) \cdot \mathbf{G}_i(\mathbf{x}, \mathbf{z}) ds_x \\
&= a_\epsilon^2 \int_S (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \cdot \mathbf{G}_i(a_\epsilon \mathbf{y} + \mathbf{x}_c, \mathbf{z}) ds_y \\
&= a_\epsilon^2 \int_S (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \cdot \mathbf{G}_i(\mathbf{x}_c, \mathbf{z}) ds_y \\
&\quad + a_\epsilon^3 \int_S (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \cdot \nabla_x \mathbf{G}_i(\mathbf{x}_c, \mathbf{z}) \mathbf{y} ds_y + \mathcal{O}(a_\epsilon^4) \\
&= a_\epsilon^3 \int_S (\lambda \kappa \mathbf{n}_y - \nabla_{s,y} \lambda) \cdot \nabla_x \mathbf{G}_i(\mathbf{x}_c, \mathbf{z}) \mathbf{y} ds_y + \mathcal{O}(a_\epsilon^4) \\
&= -a_\epsilon^3 \int_S \lambda \nabla_x \mathbf{G}_i(\mathbf{x}_c, \mathbf{z}) \mathbf{n}_y \cdot \mathbf{n}_y ds_y + \mathcal{O}(a_\epsilon^4),
\end{aligned} \tag{2.8.10}$$

where we used an expansion on \mathbf{G}_i and formula (2.2.4). To compute the term over the droplet T^ϵ we define the following,

$$\mathbf{R}_\epsilon(\mathbf{y}) = \mathbf{v}^\epsilon(a_\epsilon \mathbf{y} + \mathbf{x}_c) - \mathbf{v}(a_\epsilon \mathbf{y} + \mathbf{x}_c) - a_\epsilon \mathbf{W}(\mathbf{y}), \quad \mathbf{r}_\epsilon(\mathbf{x}) = \mathbf{R}_\epsilon\left(\frac{\mathbf{x} - \mathbf{x}_c}{a_\epsilon}\right) = \mathbf{R}_\epsilon(\mathbf{y}).$$

Thus,

$$\begin{aligned}
&\int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x(\mathbf{v}^\epsilon) d\mathbf{x} \\
&= \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x(\mathbf{r}^\epsilon(\mathbf{x})) d\mathbf{x} \\
&\quad + \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x\left(\mathbf{v}(\mathbf{x}) + a_\epsilon \mathbf{W}\left(\frac{\mathbf{x} - \mathbf{x}_c}{a_\epsilon}\right)\right) d\mathbf{x}.
\end{aligned}$$

The first integral above, in view of LEMMA 2.8.2, becomes

$$\begin{aligned}
& \left| \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x(\mathbf{r}^\epsilon(\mathbf{x})) d\mathbf{x} \right| \\
&= a_\epsilon^{n-1} \left| \int_T 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(a_\epsilon \mathbf{y} + \mathbf{x}_c, \mathbf{z}) : e_y(\mathbf{R}^\epsilon(\mathbf{y})) d\mathbf{y} \right| \\
&\leq C a_\epsilon^2 a_\epsilon^{3/2} = C a_\epsilon^{3+\frac{1}{2}}.
\end{aligned} \tag{2.8.11}$$

For the second integral we have,

$$\begin{aligned}
& \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x \left(\mathbf{v}(\mathbf{x}) + a_\epsilon \mathbf{W} \left(\frac{\mathbf{x} - \mathbf{x}_c}{a_\epsilon} \right) \right) d\mathbf{x} \\
&= 2(\mu_2 - \mu_1) a_\epsilon^n \int_T e_x(\mathbf{G}_i)(a_\epsilon \mathbf{y} + \mathbf{x}_c, \mathbf{z}) : (e_x(\mathbf{v})(a_\epsilon \mathbf{y} + \mathbf{x}_c) + e_y(\mathbf{W})(\mathbf{y})) d\mathbf{y} \\
&= 2(\mu_2 - \mu_1) a_\epsilon^n \int_T e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \{e_x(\mathbf{v})(\mathbf{x}_c) + e_y(\mathbf{W})(\mathbf{y})\} d\mathbf{y} + \mathcal{O}(a_\epsilon^4).
\end{aligned} \tag{2.8.12}$$

To complete the proof, we write $\mathbf{W}(\mathbf{y})$ as a linear combination of $\hat{\boldsymbol{\phi}}^{kl}$, solution to (2.8.5), the following way,

$$\mathbf{W}(\mathbf{y}) = \sum_{k,l=1}^3 e_{xkl}(\mathbf{v})(\mathbf{x}_c) \hat{\boldsymbol{\phi}}^{kl}(\mathbf{y}).$$

Replacing $\mathbf{W}(\mathbf{y})$ in (2.8.12) we get,

$$\begin{aligned}
& \int_{T^\epsilon} 2(\mu_2 - \mu_1) e_x(\mathbf{G}_i)(\mathbf{x}, \mathbf{z}) : e_x \left(\mathbf{v}(\mathbf{x}) + a_\epsilon \mathbf{W} \left(\frac{\mathbf{x} - \mathbf{x}_c}{a_\epsilon} \right) \right) d\mathbf{x} \\
&= 2(\mu_2 - \mu_1) a_\epsilon^3 \int_T e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \left\{ e_x(\mathbf{v})(\mathbf{x}_c) + \sum_{k,l=1}^3 e_{xkl}(\mathbf{v})(\mathbf{x}_c) e_y(\hat{\boldsymbol{\phi}}^{kl})(\mathbf{y}) \right\} d\mathbf{y} \\
&+ \mathcal{O}(a_\epsilon^4).
\end{aligned}$$

Combining the above result, (2.8.9), (2.8.10), (2.8.11), (2.8.12), and substituting $\hat{\boldsymbol{\phi}}^{kl}(\mathbf{y}) = \boldsymbol{\phi}^{kl}(\mathbf{y}) - \mathbf{B}^{kl}$ we get

$$\begin{aligned}
(v_i^\epsilon - v_i)(\mathbf{z}) &= 2(\mu_2 - \mu_1) a_\epsilon^n e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \int_T \left\{ \sum_{k,l=1}^3 e_{xkl}(\mathbf{v})(\mathbf{x}_c) e_y(\boldsymbol{\phi}^{kl})(\mathbf{y}) \right\} d\mathbf{y} \\
&- a_\epsilon^3 e_x(\mathbf{G}_i)(\mathbf{x}_c, \mathbf{z}) : \int_S \lambda(\mathbf{y}) \mathbf{n}_y \otimes \mathbf{n}_y ds_y + \mathcal{O} \left(a_\epsilon^{3+\frac{1}{2}} \right).
\end{aligned}$$

□

2.9 Asymptotics of the viscosity for emulsions with spherical droplets

This section demonstrates that the kinematic condition, given by (2.2.1d), is essential to obtaining the proper solution to problem (2.2.1). The use of the kinematic condition leads to the recovery of the proper effective viscosity given by Taylor's formula (1.1.1). The results of this section rely on the work of [23] and [33] to compute the effective viscosity of the emulsion. We begin by defining the following tensors,

$$A_{ijk\ell} = \frac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk}) - \frac{1}{3}\delta_{ij}\delta_{k\ell}, \quad B_k^{ij} = \frac{1}{2}(y_j\delta_{ik} + y_i\delta_{jk}) - \frac{1}{3}\delta_{ij}y_k$$

We remark that from a simple computation one can obtain that $e_{ij}(\mathbf{B}^{k\ell}) = A_{ijk\ell}$.

In periodic homogenization, the local problem for the suspension of spherical droplets of viscosity μ_1 in a viscous fluid of viscosity μ_2 has the following form [33]:

$$\begin{aligned} -\operatorname{div}_y(\tau^{ij}) &= \mathbf{0} && \text{in } Y - S, \\ \operatorname{div}_y \mathbf{v}^{ij} &= 0 && \text{in } Y, \\ \llbracket \mathbf{v}^{ij} \rrbracket &= \mathbf{0} && \text{on } S, \\ (B_m^{ij} + v_m^{ij}) n_m &= \left(\frac{1}{B} \int_B v_m^{ij} d\mathbf{y} \right) n_m && \text{on } S, \\ \llbracket \tau^{ij} \mathbf{n} \rrbracket &= \llbracket \tau^{ij} \mathbf{n} \rrbracket \cdot \mathbf{n} \mathbf{n} && \text{on } S, \\ \mathbf{v}^{ij} &\text{ is } Y\text{-periodic,} \\ \int_{\partial B} \llbracket \tau^{ij} \mathbf{n} \rrbracket ds_y &= \mathbf{0} \end{aligned} \tag{2.9.1}$$

where Y is the unit period cell, S is the surface of the spherical drop, and

$$\tau_{ml}^{ij} = 2\mu(A_{ijml} + e_{ml}(v^{ij})) - p^{ij}\delta_{lm}$$

We remark that since the shape of the drop has already been determined, one cannot impose the normal stress jump.

We define the Hilbert space $V_{per} = \{\mathbf{w} \in H_{per}^1(Y)^3 \mid \operatorname{div} \mathbf{w} = 0\} / \mathbb{R}$. Moreover, we define the closed, convex, and unbounded subset of V_{per} by

$$H^{ij} = \left\{ \mathbf{w} \in V_{per} \mid (B_m^{ij} + w_m) n_m = \left(\frac{1}{|B|} \int_B w_m d\mathbf{y} \right) n_m \text{ on } S \right\} / \mathbb{R}.$$

The weak formulation of (2.9.1) yields the variational inequality: Find $\mathbf{v}^{ij} \in V_{per}$ such that

$$\int_Y 2\mu e_{kl}(\mathbf{B}^{ij} + \mathbf{v}^{ij}) e_{kl}(\boldsymbol{\phi} - \mathbf{v}^{ij}) d\mathbf{y} = 0 \text{ for any } \boldsymbol{\phi} \in H^{ij} \tag{2.9.2}$$

for which local existence and uniqueness results can be found in [30].

According to [33], the formula for the homogenized coefficients is

$$2\mu_{ijkl}^H = \frac{1}{|Y|} \int_Y (\tau_{kl}^{ij} - \frac{1}{3} \delta_{kl} \tau_{pp}^{ij}) d\mathbf{y} - \int_S \left[\frac{1}{2} (\tau_{kq}^{ij} y_l n_q + \tau_{lq}^{ij} y_k n_q) - \frac{1}{3} \delta_{kl} \tau_{pq}^{ij} y_p n_q \right] ds_y \quad (2.9.3)$$

Using the divergence free feature of the vector field $\mathbf{B}^{kl} + \mathbf{v}^{kl}$, we can re-write (2.9.3) as follows:

$$\begin{aligned} 2\mu_{ijkl}^H |Y| &= \int_Y (\tau_{kl}^{ij} - \frac{1}{3} \delta_{kl} \tau_{pp}^{ij}) d\mathbf{y} - \int_S \left[\frac{1}{2} (\tau_{kq}^{ij} y_l n_q + \tau_{lq}^{ij} y_k n_q) - \frac{1}{3} \delta_{kl} \tau_{pq}^{ij} y_p n_q \right] ds_y \\ &= \int_Y (\tau_{kl}^{ij} - \frac{1}{3} \delta_{kl} \tau_{pp}^{ij}) d\mathbf{y} - \int_S \left[\tau_{pq}^{ij} n_q \right] B_p^{kl} ds_y \\ &= \int_Y (\tau_{kl}^{ij} - \frac{1}{3} \delta_{kl} \tau_{pp}^{ij}) d\mathbf{y} - \int_S \left[\tau_{\alpha\beta}^{ij} n_\beta \right] n_\alpha B_p^{kl} n_p ds_y \\ &= \int_Y \tau_{pq}^{ij} e_{pq}(\mathbf{B}^{kl} + \mathbf{v}^{kl}) d\mathbf{y} = \int_Y 2\mu e_{pq}(\mathbf{B}^{ij} + \mathbf{v}^{ij}) e_{pq}(\mathbf{B}^{kl} + \mathbf{v}^{kl}) d\mathbf{y} \end{aligned}$$

In the next subsection we compute the solution of the local problem explicitly.

2.9.1 Flow of concentric liquid spheres with fixed center of mass

Define the sphere centered at zero with radius b to be $B(0, b)$. Let a be a fixed number such that $a < b$ and define $\lambda = a/b$, $\eta = \mu_1/\mu_2$ and consider the local problem (2.9.1) with Dirichlet boundary condition on the exterior,

$$\begin{aligned} -\operatorname{div}_y(\tau^{ij}) &= \mathbf{0} && \text{in } B(0, b) - \partial B(0, a), \\ \operatorname{div}_y \mathbf{v}^{ij} &= 0 && \text{in } B(0, b), \\ \llbracket \mathbf{v}^{ij} \rrbracket &= \mathbf{0} && \text{on } \partial B(0, a), \\ (B_m^{ij} + v_m^{ij}) n_m &= \left(\frac{1}{B} \int_B v_m^{ij} d\mathbf{y} \right) n_m && \text{on } \partial B(0, a), \\ \llbracket \tau^{ij} \mathbf{n} \rrbracket &= \llbracket \tau^{ij} \mathbf{n} \rrbracket \cdot \mathbf{n} \mathbf{n} && \text{on } \partial B(0, a), \\ \mathbf{v}^{ij} &= \mathbf{0} && \text{on } \partial B(0, b), \\ \int_{\partial B(0, a)} \llbracket \tau^{ij} \mathbf{n} \rrbracket ds_y &= \mathbf{0} \end{aligned} \quad (2.9.4)$$

Define $\mathbf{u}^{ij} = \mathbf{v}^{ij} + \mathbf{B}^{ij}$. Then \mathbf{u}^{ij} satisfies the following problem,

$$\begin{aligned}
-\operatorname{div} \sigma^{ij} &= \mathbf{0} && \text{in } B(0, b) \setminus \partial B(0, a) \\
\sigma_{k\ell}^{ij} &= -p^{ij} \delta_{k\ell} + 2\mu e_{k\ell}(\mathbf{u}^{ij}) \\
\operatorname{div} \mathbf{u}^{ij} &= 0 && \text{in } B(0, b) \\
\llbracket \mathbf{u}^{ij} \rrbracket &= \mathbf{0} && \text{on } \partial B(0, a) \\
\mathbf{u}^{ij} \cdot \mathbf{n} &= \mathbf{0} && \text{on } \partial B(0, a) \\
\llbracket \mathbf{n} \times \sigma^{ij} \mathbf{n} \rrbracket &= 0 && \text{on } \partial B(0, a) \\
\mathbf{u}^{ij} &= -\mathbf{B}^{ij} && \text{on } \partial B(0, b)
\end{aligned} \tag{2.9.5}$$

Writing the problem in this form assumes that the spherical droplets have a fixed centers of mass and thus no forces or torques are exerted. The solution to (2.9.5) is given by [23] to be,

$$\begin{aligned}
\hat{u}_k^{ij} &= \frac{\Phi}{\Delta} \left[- \left(\frac{-5r^2}{a^2} + 3r \right) B_k^{ij} - \frac{2}{a^2} B_\ell^{ij} \mathbf{x}_\ell \mathbf{x}_k \right] && \text{in } 0 \leq r \leq a \\
u_k^{ij} &= \frac{1}{\Delta} \left[- \left(\phi_1 - 5r^2 \phi_2 \frac{1}{a^2} + 4a^5 \phi_3 \frac{1}{r^5} \right) B_k^{ij} \right. \\
&\quad \left. - \left(2\phi_2 \frac{1}{a^2} + 2a^3 \phi_4 \frac{1}{r^5} - 10a^5 \phi_3 \frac{1}{r^7} \right) B_k^{ij} \right] && \text{in } a \leq r \leq b
\end{aligned}$$

Here,

$$\Phi = 5\lambda^7 - 7\lambda^5 + 2,$$

$$\phi_1 = 5(5\eta - 2)\lambda^7 - 21\eta\lambda^5 - 4\eta - 4,$$

$$\phi_2 = \lambda^5(5\eta\lambda^2 - 5\eta - 2),$$

$$\phi_3 = (1 - \eta)\lambda^5 + \eta,$$

$$\phi_4 = 5(1 - \eta)\lambda^7 + 5\eta + 2,$$

$$\Delta = (4(1 - \eta))\lambda^{10} - (5(2 - 5\eta))\lambda^7 - 42\eta\lambda^5 + (5(2 + 5\eta))\lambda^3 - 4 - 4\eta.$$

Using formula (2.9.3) we can compute the homogenized viscosity to be,

$$\begin{aligned}
2 \mu_{ijkl}^H |Y| &= \int_Y 2 \mu e_{pq}(\mathbf{u}^{ij}) e_{pq}(\mathbf{u}^{kl}) d\mathbf{y} \\
&= \int_{Y-B(0,b)} 2 \mu_2 e_{pq}(\mathbf{B}^{ij}) e_{pq}(\mathbf{u}^{kl}) d\mathbf{y} + \int_{B(0,b)} 2 \mu e_{pq}(\mathbf{u}^{ij}) e_{pq}(\mathbf{u}^{kl}) d\mathbf{y}.
\end{aligned}$$

The integrand in the first integral is a constant and can we immediately compute its value to be $2 \mu_2 A_{ijkl} (8b^3 - \frac{4}{3} \pi b^3)$. For the second integral, since we have explicit solutions of the velocity vector field, we can compute the integral using **MAPLE**. Thus,

$$\begin{aligned}
2 \mu_{ijkl}^H &= 2 \mu_2 A_{ijkl} \left(1 - \frac{\pi}{6}\right) \\
&+ A_{ijkl} \mu_2 \frac{\pi}{3} \left(\frac{6(1-\eta)\lambda^{10} - 5(5\eta-2)\lambda^7 + 42\eta\lambda^5 - 3(5\eta+2)\lambda^3 + 4(\eta+1)}{4(1-\eta)\lambda^{10} + 5(5\eta-2)\lambda^7 + 42\eta\lambda^5 - 5(5\eta+2)\lambda^3 + 4(1+\eta)} \right).
\end{aligned}$$

Isolating the fraction of the second term we have,

$$\begin{aligned}
&(4(\eta+1) - 3(5\eta+2)\lambda^3 + 42\eta\lambda^5 - 5(5\eta-2)\lambda^7 + 6(1-\eta)\lambda^{10}) \\
&\times \frac{1}{4(1+\eta)} \Big/ \left(1 - \frac{4(\eta-1)\lambda^{10} - 5(5\eta-2)\lambda^7 - 42\eta\lambda^5 + 5(5\eta+2)\lambda^3}{4(1+\eta)}\right).
\end{aligned}$$

When λ is small, we can use a geometric series expansion to obtain

$$\begin{aligned}
&(4(\eta+1) - 3(5\eta+2)\lambda^3 + 42\eta\lambda^5 - 5(5\eta-2)\lambda^7 + 6(1-\eta)\lambda^{10}) \\
&\times \frac{1}{4(1+\eta)} \left(1 + \sum_{k=1}^{\infty} \left[\frac{4(\eta-1)\lambda^{10} - 5(5\eta-2)\lambda^7 - 42\eta\lambda^5 + 5(5\eta+2)\lambda^3}{4(1+\eta)} \right]^k \right) \\
&= \left(1 - \frac{3(5\eta+2)}{4(1+\eta)}\lambda^3 + \frac{42\eta}{4(1+\eta)}\lambda^5 - \frac{5(5\eta-2)}{4(1+\eta)}\lambda^7 + \frac{6(1-\eta)}{4(1+\eta)}\lambda^{10}\right) \\
&\times \left(1 + \sum_{k=1}^{\infty} \left[\frac{4(\eta-1)\lambda^{10} - 5(5\eta-2)\lambda^7 - 42\eta\lambda^5 + 5(5\eta+2)\lambda^3}{4(1+\eta)} \right]^k \right).
\end{aligned}$$

Keeping only the terms that involve λ^3 and λ^6 we have

$$\begin{aligned}
2 \mu_{ijkl}^H &= 2 \mu_2 A_{ijkl} \left(1 - \frac{\pi}{6}\right) \\
&+ 2 \mu_2 A_{ijkl} \frac{\pi}{6} \left(1 + \frac{(5\eta+2)}{2(1+\eta)}\lambda^3 + \frac{25}{2} \left(\frac{5(5\eta+2)}{2(1+\eta)}\right)^2 \lambda^6 + \dots\right). \tag{2.9.6}
\end{aligned}$$

If we define by ϕ the concentration of the spherical droplets, then

$$\phi = \frac{(4/3) \pi a^3}{8 b^3} = \frac{\pi}{6} \lambda^3$$

Therefore, for low concentrations (i.e. λ small) and up to first order we have,

$$\mu_{ijkl}^H = \mu_2 A_{ijkl} \left(1 + \frac{(5\eta + 2)}{2(1 + \eta)} \phi + \mathcal{O}(\phi^2) \right) \quad (2.9.7)$$

Hence, the *effective scalar viscosity* μ^* , is the viscosity of the Newtonian fluid that behaves as the mixture of arbitrarily oriented spherical droplets of viscosity μ_1 , that are invariant under suspension rotation. The effective scalar viscosity is computed as the angular averaging of the tensor in (2.9.7)

$$\mu^* = \mu_2 \left(1 + \frac{5\eta + 2}{2(1 + \eta)} \phi + \mathcal{O}(\phi^2) \right) \quad (2.9.8)$$

which agrees with Taylor's result [57].

2.10 Conclusions

We consider an emulsion formed by two Newtonian fluids, one being dispersed in the other under the form of droplets, in the presence of non-uniform surface tension. We investigate the dilute case where the droplet size a_ϵ is much smaller than the distance ϵ between the droplets' centers. We prove using Γ -convergence in THEOREM 2.5.1 that the limit behavior when $\epsilon \rightarrow 0$ is described by the unperturbed Stokes flow and in THEOREM 2.7.3 we estimate the order of convergence of the velocity in the H^1 norm to be $\mathcal{O}(a_\epsilon^{3/2})$. In the THEOREM 2.8.3 we improve the convergence result and determine the first corrector in the velocity expansion in the L^2 norm.

Similar results were obtained in [6]; however, in this chapter we take into account the kinematic condition (2.2.1d) at the droplet surface, which insures that the droplet boundary is a material boundary. Only with this condition one can recover Taylor's [57] and Einstein's [16], [17] viscosity formulas for spherical fluid drops and spherical rigid particles respectively.

Our results are also not limited to uniform surface tension.

Chapter 3

Multiscale analysis of magnetorheological fluids

Introduction

Magneto-rheological (MR) fluids are a suspension of non-colloidal, ferromagnetic particles in a non-magnetizable carrier fluid. The particles are often of micron size ranging anywhere from $0.05 - 10 \mu m$ with particle volume fraction from $10 - 40 \%$. They were discovered by J. Rabinow in 1948 [49]. Around the same time W. Winslow in 1939 discovered electrorheological (ER) fluids, a closely related counterpart. In elasticity similar material are referred to as magneto-elastic composites [48]. Such a case will not be treated here. The interested reader can consult [7] for more details.

MR fluids respond to an external magnetic field by a rapid, reversible change in their properties. They can transform from a liquid to a semi solid state in a matter of milliseconds. Upon the application of a magnetic field, the dipole interaction of adjacent particles aligns the particles in the direction of the magnetic field lines. Namely particles attract one another along the magnetic field lines and repel one another in the direction perpendicular to them. This leads to the formation of aggregate structures. Once these aggregate structures are formed, the MR fluid exhibits a yield stress that is dependent and controlled by the applied external magnetic field [34], [21].

The formation of these aggregates means that the behavior of the fluid is non-Newtonian. In many works, the Bingham constitutive law is used as an approximation to model the response of the MR and ER fluids, particularly in shear experiments [47], [20], [22]. Although the Bingham model has proven itself useful in characterizing the behavior of MR fluids, it is not always sufficient. Recent experimental data show that true MR fluids exhibit departures from the Bingham model [61], [22].

Another member of the magnetic suspensions family are ferrofluids. Ferrofluids are stable colloidal suspensions of nanoparticles in a non-magnetizable carrier fluid. The initiation into the hydrodynamics of ferrofluids began with Neuringer and Rosensweig in 1964 [46] and by a series of works by Rosensweig and co-workers summarized in [50]. The model introduced

in [46] assumes that the magnetization is collinear with the magnetic field and has been very useful in describing quasi-stationary phenomena. This work was extended by Shliomis [54] by avoiding the collinearity assumption of the magnetization and the magnetic field and by considering the rotation of the nanoparticles with respect to the fluid they are suspended in.

The models mentioned above have all been derived phenomenologically. The first attempt to use homogenization mechanics to describe the behavior of MR \ ER fluids was carried out in [26], [27] and [47]. In the works [26], [27] the influence of the external magnetic field is introduced as a volumic density force acting on each particle and as a surface density force acting on the boundary of each particle. The authors in [47] extend the work in [27], for ER fluids, by presenting a more complete model that couples the conservation of mass and momentum equations with Maxwell's equations through the Maxwell stress tensor. As an application they consider a uniform shearing of the ER fluid submitted to a uniform electric field boundary conditions in a two dimensional slab and they recover that the stress tensor at the macroscopic scale has exactly the form of the Bingham constitutive equation.

The authors in [47], [51], [50] use models that decouple the conservation of mass and momentum equations from the Maxwell equations. Thus in principle one can solve the Maxwell equations and use the resulting magnetic or electric field as a force in the conservation of mass and momentum equations.

The present work focuses on a suspension of rigid magnetizable particles in a Newtonian viscous fluid with an applied external magnetic field. We assume the fluid to be electrically non-conducting. Thus, we use the quasi-static Maxwell equations coupled with the Stokes equations through Ohm's law to capture the magnetorheological effect. In doing so we extend the model of [47]. Thus the Maxwell and the balance of mass and momentum equations must be simultaneously solved.

In section 3.1 we introduce the problem in the periodic homogenization framework. The particles are periodically distributed and the size of the period is of the same order as the characteristic length of the particles. We assume the fluid velocity is continuous across the particle interface and that the particles are in equilibrium in the presence of the magnetic field.

The two scale expansion is carried out in section 3.2 where we obtain a decoupled set of problems at order $\mathcal{O}(\epsilon^{-1})$.

In section 3.3 and in section 3.4 we study the local problems that arise from the contribution of the bulk magnetic field as well as the bulk velocity and provide new constitutive laws for Maxwell's equations.

In section 3.5 we provide the governing effective equations of the MR fluid which include, in addition to the viscous stresses, a "Maxwell type" stress. Furthermore, we provide formulas for the effective viscosity and effective coefficients for the Maxwell type stress that generalize those in [27].

Section 3.6 is devoted to comparing the results of the proposed model against experimental data. We compute the constitutive coefficients for an aqueous MR fluid with magnetite particles using the finite element method, we obtain the velocity profiles of both Poiseuille and Couette flows for this MR fluid and plot the stress vs shear rate curve for different values

of the applied magnetic field, that exhibit a yield stress comparable to the one obtained in experiments (e.g. [61])

3.1 Problem statement

For the homogenization setting of the suspension problem we define $\Omega \subset \mathbb{R}^3$, to be a bounded open set with sufficiently smooth boundary $\partial\Omega$, $Y = \left(-\frac{1}{2}, \frac{1}{2}\right)^3$ be the unit cube in \mathbb{R}^3 , and \mathbb{Z}^3 is the set of all three dimensional vectors with integer components. For every positive ϵ , let N^ϵ be the set of all points $\ell \in \mathbb{Z}^3$ such that $\epsilon(\ell + Y)$ is strictly included in Ω and denote by $|N^\epsilon|$ their total number. Let T be the closure of an open connected set with sufficiently smooth boundary, compactly included in Y . For every $\epsilon > 0$ and $\ell \in N^\epsilon$ we consider the set $T_\ell^\epsilon \subset \subset \epsilon(\ell + Y)$, where $T_\ell^\epsilon = \epsilon(\ell + T)$. The set T_ℓ^ϵ represents one of the solid particles suspended in the fluid, and $S_\ell^\epsilon = \partial T_\ell^\epsilon$ denotes its surface (see FIGURE 3.1). We now define the following subsets of Ω :

$$\Omega_{1\epsilon} = \bigcup_{\ell \in N^\epsilon} T_\ell^\epsilon, \quad \Omega_{2\epsilon} = \Omega \setminus \overline{\Omega_{1\epsilon}},$$

where $\Omega_{1\epsilon}$ is the domain occupied by the rigid particles, and $\Omega_{2\epsilon}$ is the domain occupied by the surrounding fluid of viscosity ν . By \mathbf{n} we indicate the unit normal on the particle surface pointing outwards and by $[[\cdot]]$ we indicate the jump discontinuity between the fluid and the rigid part.

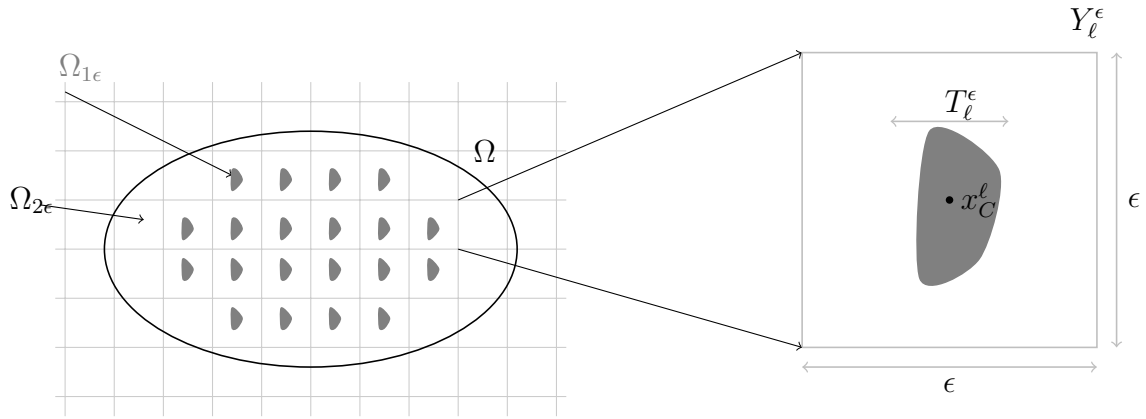


Figure 3.1: Schematic of the periodic suspension of rigid magnetizable particles in non-magnetizable fluid

The description of the problem is,

$$\rho \frac{\partial \mathbf{v}^\epsilon}{\partial t} + \rho (\mathbf{v}^\epsilon \cdot \nabla) \mathbf{v}^\epsilon - \operatorname{div} \sigma^\epsilon = \rho \mathbf{f}, \text{ where } \sigma^\epsilon = 2\nu e(\mathbf{v}^\epsilon) - p^\epsilon I \quad \text{in } \Omega_{2\epsilon}, \quad (3.1.1a)$$

$$\operatorname{div} \mathbf{v}^\epsilon = 0, \quad \operatorname{div} \mathbf{B}^\epsilon = 0, \quad \operatorname{curl} \mathbf{H}^\epsilon = \mathbf{0} \quad \text{in } \Omega_{2\epsilon}, \quad (3.1.1b)$$

$$e(\mathbf{v}^\epsilon) = 0, \quad \operatorname{div} \mathbf{B}^\epsilon = 0, \quad \operatorname{curl} \mathbf{H}^\epsilon = \eta \mathbf{v}^\epsilon \times \mathbf{B}^\epsilon \quad \text{in } \Omega_{1\epsilon}, \quad (3.1.1c)$$

where $\mathbf{B}^\epsilon = \mu^\epsilon \mathbf{H}^\epsilon$ and with boundary conditions on the surface of each particle T_ℓ^ϵ ,

$$\begin{aligned} \llbracket \mathbf{v}^\epsilon \rrbracket &= \mathbf{0}, \quad \llbracket \mathbf{B}^\epsilon \cdot \mathbf{n} \rrbracket = 0, \quad \llbracket \mathbf{n} \times \mathbf{H}^\epsilon \rrbracket = \mathbf{0} && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{0}, \quad \mathbf{H}^\epsilon = \mathbf{b} && \text{on } \partial\Omega, \end{aligned} \quad (3.1.2)$$

where ρ is the density of the fluid, \mathbf{v}^ϵ represents the velocity field, p^ϵ the pressure, $e(\mathbf{v}^\epsilon)$ the strain rate, \mathbf{f} the body forces, \mathbf{n} the exterior normal to the particles, \mathbf{H}^ϵ the magnetic field, μ^ϵ is the magnetic permeability of the material, $\mu^\epsilon(\mathbf{x}) = \mu_1$ if $\mathbf{x} \in \Omega_{1\epsilon}$ and $\mu^\epsilon(\mathbf{x}) = \mu_2$ if $\mathbf{x} \in \Omega_{2\epsilon}$, η the electric conductivity of the rigid particles, and \mathbf{b} is an applied constant magnetic field on the exterior boundary of the domain Ω .

In the absence of a magnetic field, the rigid particles suspended in the fluid align themselves in the direction of the flow. However, when the MR fluid is submitted to a magnetic field, the rigid particle are subjected to a force that makes them behave like a dipole aligned in the direction of the magnetic field. This force can be written in the form,

$$\mathbf{F}^\epsilon = -\frac{1}{2} |\mathbf{H}^\epsilon|^2 \nabla \mu^\epsilon$$

where $|\cdot|$ represents the standard Euclidean norm. The force can be written in terms of the Maxwell stress $\tau_{ij}^\epsilon = \mu^\epsilon H_i^\epsilon H_j^\epsilon - \frac{1}{2} \mu^\epsilon H_k^\epsilon H_k^\epsilon \delta_{ij}$ as,

$$\mathbf{F}^\epsilon = \operatorname{div} \tau^\epsilon + \mathbf{B}^\epsilon \times \operatorname{curl} \mathbf{H}^\epsilon.$$

Since the magnetic permeability is considered constant in each phase, the force is zero in each phase as well. Therefore, we deduce that

$$\operatorname{div} \tau^\epsilon = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_{2\epsilon} \\ -\mathbf{B}^\epsilon \times \operatorname{curl} \mathbf{H}^\epsilon & \text{if } \mathbf{x} \in \Omega_{1\epsilon}. \end{cases} \quad (3.1.3)$$

We remark that unlike the viscous stress σ^ϵ , the Maxwell stress is present in the entire domain Y . Hence, balance of forces and torques in each particle can be written as,

$$\begin{aligned} 0 &= \int_{S_\ell^\epsilon} (\sigma^\epsilon \mathbf{n} + \llbracket \tau^\epsilon \mathbf{n} \rrbracket) ds - \int_{T_\ell^\epsilon} \mathbf{B}^\epsilon \times \operatorname{curl} \mathbf{H}^\epsilon d\mathbf{x} + \int_{T_\ell^\epsilon} \rho \mathbf{f} d\mathbf{x} \\ 0 &= \int_{S_\ell^\epsilon} (\sigma^\epsilon \mathbf{n} + \llbracket \tau^\epsilon \mathbf{n} \rrbracket) \times (\mathbf{x} - \mathbf{x}_c^\ell) ds \\ &\quad - \int_{T_\ell^\epsilon} (\mathbf{B}^\epsilon \times \operatorname{curl} \mathbf{H}^\epsilon) \times (\mathbf{x} - \mathbf{x}_c^\ell) d\mathbf{x} + \int_{T_\ell^\epsilon} \rho \mathbf{f} \times (\mathbf{x} - \mathbf{x}_c^\ell) d\mathbf{x} \end{aligned} \quad (3.1.4)$$

where \mathbf{x}_c^ℓ is the center of mass of the particle T_ℓ^ϵ . Therefore the system governing the MR suspension are equations (3.1.1)–(3.1.2) together with (3.1.4).

3.1.1 Dimensional Analysis

Before we proceed further we non-dimensionalize the problem. Denote by $x^* = x/L$, $\mathbf{v}^* = \mathbf{v}/V$, $p^* = p/\nu_L^V$, $\mathbf{H}^* = \mathbf{H}/H$ and $\mathbf{f}^* = \mathbf{f}/\frac{V^2}{L}$, and $\mu^{\epsilon*} = \mu^\epsilon/\mu_2$. Substituting the above expressions into (3.1.1) as well as in the balance of forces and torques, and using the fact that the flow is assumed to be at low Reynolds numbers, we obtain

$$\begin{aligned} Re \left(\frac{\partial \mathbf{v}^{\epsilon*}}{\partial t} + (\mathbf{v}^{\epsilon*} \cdot \nabla) \mathbf{v}^{\epsilon*} \right) - \operatorname{div}^* \sigma^{\epsilon*} &= Re \mathbf{f}^*, \quad \text{where } \sigma^{\epsilon*} = 2e(\mathbf{v}^{\epsilon*}) - p^{\epsilon*} I && \text{in } \Omega_{2\epsilon}, \\ \operatorname{div}^* \mathbf{v}^{\epsilon*} &= 0, \quad \operatorname{div}^* \mathbf{B}^{\epsilon*} = 0, \quad \operatorname{curl}^* \mathbf{H}^{\epsilon*} = \mathbf{0} && \text{in } \Omega_{2\epsilon}, \\ e^*(\mathbf{v}^{\epsilon*}) &= 0, \quad \operatorname{div}^* \mathbf{B}^{\epsilon*} = 0, \quad \operatorname{curl}^* \mathbf{H}^{\epsilon*} = R_m \mathbf{v}^{\epsilon*} \times \mathbf{B}^{\epsilon*} && \text{in } \Omega_{1\epsilon}, \end{aligned}$$

where $\mathbf{B}^{\epsilon*} = \mu^{\epsilon*} \mathbf{H}^{\epsilon*}$ and with boundary conditions on the surface of each particle T_ℓ^ϵ ,

$$\begin{aligned} \llbracket \mathbf{v}^{\epsilon*} \rrbracket &= \mathbf{0}, \quad \llbracket \mathbf{B}^{\epsilon*} \cdot \mathbf{n} \rrbracket = 0, \quad \llbracket \mathbf{n} \times \mathbf{H}^{\epsilon*} \rrbracket = \mathbf{0} && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^{\epsilon*} &= \mathbf{0}, \quad \mathbf{H}^{\epsilon*} = \mathbf{b}_* && \text{on } \partial\Omega. \end{aligned}$$

together with the balance of forces and torques,

$$\begin{aligned} 0 &= \int_{S_\ell^\epsilon} \sigma^{\epsilon*} \mathbf{n} ds^* + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^{\epsilon*} \mathbf{n} \rrbracket ds^* - \alpha \int_{T_\ell^\epsilon} \mathbf{B}^{\epsilon*} \times \operatorname{curl}^* \mathbf{H}^{\epsilon*} d\mathbf{x}^* + Re \int_{T_\ell^\epsilon} \mathbf{f}^* d\mathbf{x}^* \\ 0 &= \int_{S_\ell^\epsilon} \sigma^{\epsilon*} \mathbf{n} \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) ds^* + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^{\epsilon*} \mathbf{n} \rrbracket \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) ds^* \\ &\quad - \alpha \int_{T_\ell^\epsilon} (\mathbf{B}^{\epsilon*} \times \operatorname{curl}^* \mathbf{H}^{\epsilon*}) \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) d\mathbf{x}^* + Re \int_{T_\ell^\epsilon} \mathbf{f}^* \times (\mathbf{x}^* - \mathbf{x}_c^{\ell*}) d\mathbf{x}^* \end{aligned}$$

where $Re = \frac{\rho V L}{\nu}$ is the Reynolds number, $\alpha = \frac{\mu_2 H^2 L}{\nu V}$ is the Alfven number, and $R_m = \eta \mu_1 L V$ is the magnetic Reynolds number.

In what follows we drop the star for simplicity. Moreover, for low Reynolds numbers the preceding equations become,

$$-\operatorname{div} \sigma = \mathbf{0}, \quad \text{where } \sigma = 2e(\mathbf{v}) - pI \quad \text{in } \Omega_{2\epsilon}, \quad (3.1.5a)$$

$$\operatorname{div} \mathbf{v} = 0, \quad \operatorname{div} \mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} = \mathbf{0} \quad \text{in } \Omega_{2\epsilon}, \quad (3.1.5b)$$

$$e(\mathbf{v}) = 0, \quad \operatorname{div} \mathbf{H} = 0, \quad \operatorname{curl} \mathbf{H} = R_m \mathbf{v} \times \mathbf{B} \quad \text{in } \Omega_{1\epsilon}, \quad (3.1.5c)$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^\epsilon \rrbracket &= \mathbf{0}, \quad \llbracket \mathbf{B}^\epsilon \cdot \mathbf{n} \rrbracket = 0, \quad \llbracket \mathbf{n} \times \mathbf{H}^\epsilon \rrbracket = \mathbf{0} && \text{on } S_\ell^\epsilon, \\ \mathbf{v}^\epsilon &= \mathbf{0}, \quad \mathbf{H}^\epsilon = \mathbf{b} && \text{on } \partial\Omega, \end{aligned} \quad (3.1.6)$$

together with the balance of forces and torques,

$$\begin{aligned} 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n} \, ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n} \rrbracket \, ds - \alpha \int_{T_\ell^\epsilon} \mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon \, d\mathbf{x} \\ 0 &= \int_{S_\ell^\epsilon} \sigma^\epsilon \mathbf{n} \times (\mathbf{x} - \mathbf{x}_c^\ell) \, ds + \alpha \int_{S_\ell^\epsilon} \llbracket \tau^\epsilon \mathbf{n} \rrbracket \times (\mathbf{x} - \mathbf{x}_c^\ell) \, ds \\ &\quad - \alpha \int_{T_\ell^\epsilon} (\mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon) \times (\mathbf{x} - \mathbf{x}_c^\ell) \, d\mathbf{x} \end{aligned} \quad (3.1.7)$$

In the next section we will use a two scale expansion the velocity, pressure and the magnetic field.

3.2 Two scale expansions

Using a two scale expansion on \mathbf{v}^ϵ , \mathbf{H}^ϵ and p^ϵ we have,

$$\mathbf{v}^\epsilon(\mathbf{x}) = \sum_{i=0}^{+\infty} \mathbf{v}^i(\mathbf{x}, \mathbf{y}), \quad \mathbf{H}^\epsilon(\mathbf{x}) = \sum_{i=0}^{+\infty} \mathbf{H}^i(\mathbf{x}, \mathbf{y}), \quad p^\epsilon(\mathbf{x}) = \sum_{i=0}^{+\infty} p^i(\mathbf{x}, \mathbf{y}) \text{ with } \mathbf{y} = \frac{\mathbf{x}}{\epsilon}.$$

One can show that \mathbf{v}^0 is independent of \mathbf{y} and can thus obtain the following problem at order ϵ^{-1} ,

$$-\frac{\partial \sigma_{ij}^0}{\partial y_j} = 0 \quad \text{in } Y_f, \quad (3.2.1a)$$

$$\sigma_{ij}^0 = -p^0 \delta_{ij} + 2\nu (e_{ijx}(\mathbf{v}^0) + e_{ijy}(\mathbf{v}^1)) \quad (3.2.1b)$$

$$\frac{\partial v_j^0}{\partial x_j} + \frac{\partial v_j^1}{\partial y_j} = 0 \quad \text{in } Y_f, \quad (3.2.1c)$$

$$e_{ijx}(\mathbf{v}^0) + e_{ijy}(\mathbf{v}^1) = 0 \quad \text{in } T, \quad (3.2.1d)$$

$$\frac{\partial B_j^0}{\partial y_j} = 0, \quad \epsilon_{ijk} \frac{\partial H_k^0}{\partial y_j} = 0 \text{ where } B_i^0 = \mu H_i^0 \quad \text{in } Y, \quad (3.2.1e)$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^1 \rrbracket &= \mathbf{0}, \quad \llbracket \mathbf{B}^0 \cdot \mathbf{n} \rrbracket = 0, \quad \llbracket \mathbf{n} \times \mathbf{H}^0 \rrbracket = \mathbf{0} && \text{on } S, \\ \mathbf{v}^1, \quad \mathbf{H}^0 &\text{are } Y\text{-periodic.} \end{aligned} \quad (3.2.2)$$

At order of ϵ^2 and ϵ^3 we obtain the balance of forces and torques for the particle T respectively,

$$\begin{aligned} 0 &= \int_S \sigma^0 \mathbf{n} ds + \alpha \int_S \llbracket \tau^0 \mathbf{n} \rrbracket ds - \alpha \int_T \mathbf{B}^0 \times \text{curl}_Y \mathbf{H}^0 d\mathbf{y}, \\ 0 &= \int_S \mathbf{y} \times \sigma^0 \mathbf{n} ds + \alpha \int_S \mathbf{y} \times \llbracket \tau^0 \mathbf{n} \rrbracket ds - \alpha \int_T \mathbf{y} \times (\mathbf{B}^0 \times \text{curl}_Y(\mathbf{H}^0)) d\mathbf{y} \end{aligned} \quad (3.2.3)$$

where Y is the unit cell divided into two regions, the fluid region Y_f and the rigid particle T . By S we denote the surface of the rigid particle, by $\mu(\mathbf{y}) = \mu_1/\mu_2$ if $\mathbf{y} \in T$ and $\mu(\mathbf{y}) = 1$ if $\mathbf{y} \in Y_f$ the magnetic permeability in the unit cell Y , and by

$$\tau_{ij}^0 = \mu H_i^0 H_j^0 - \frac{1}{2} \mu H_k^0 H_k^0 \delta_{ij}, \quad (3.2.4)$$

we denote the Maxwell stress. We remark that since from (3.2.1e) $\text{curl}_Y(\mathbf{H}^0) = \mathbf{0}$ in Y , the balance of forces and torques simplify to the following,

$$0 = \int_S \sigma^0 \mathbf{n} + \alpha \int_S \llbracket \tau^0 \mathbf{n} \rrbracket ds \text{ and } 0 = \int_S \mathbf{y} \times \sigma^0 \mathbf{n} ds + \alpha \int_S \mathbf{y} \times \llbracket \tau^0 \mathbf{n} \rrbracket ds. \quad (3.2.5)$$

Thus, at first order, in the problem (3.2.1)-(3.2.5) the Stokes and Maxwell equations are coupled through the balance of forces and torques (3.2.5).

3.3 Constitutive relations for Maxwell's equations

3.3.1 Study of the local problem

Using the results from the two scale expansions we can see that the $\text{curl}_Y(\mathbf{H}^0) = \mathbf{0}$ in Y and thus we can write

$$H_i^0 = -\frac{\partial \psi(\mathbf{x}, \mathbf{y})}{\partial y_i} + \tilde{H}_i^0(\mathbf{x}), \quad (3.3.1)$$

where \tilde{H}_i^0 is the average of H_i^0 over the \mathbf{y} variable and $\tilde{\psi} = 0$. Using the fact $\text{div}_Y \mathbf{B}^0 = 0$ in Y , $B_i^0 = \mu H_i^0$ and the boundary conditions (3.1.6) we have,

$$\begin{aligned} -\frac{\partial}{\partial y_i} \left(\mu \left(-\frac{\partial \psi}{\partial y_i} + \tilde{H}_i^0 \right) \right) &= 0 \quad \text{in } Y, \\ \llbracket \mu \left(-\frac{\partial \psi}{\partial y_i} + \tilde{H}_i^0 \right) n_i \rrbracket &= 0 \quad \text{on } S, \\ \psi &\text{ is } Y\text{-periodic, } \tilde{\psi} = 0. \end{aligned} \quad (3.3.2)$$

Introduce

$$W_{per}(Y) = \{w \in H_{per}^1(Y) \mid \llbracket w \rrbracket = 0 \text{ on } S, \quad \tilde{w} = 0\},$$

then the variational formulation of (3.3.2) is

$$\begin{aligned} &\text{Find } \psi \in W_{per}(Y) \text{ such that} \\ &\int_Y \mu \frac{\partial \psi}{\partial y_i} \frac{\partial v}{\partial y_i} d\mathbf{y} = \tilde{H}_i^0 \int_Y \mu \frac{\partial v}{\partial y_i} d\mathbf{y}. \end{aligned} \quad (3.3.3)$$

Since we have imposed that ψ has average zero over the unit cell Y , the solution to (3.3.3) can be determined uniquely by a simple application of the Lax-Milgram lemma.

Let ϕ^k be the unique solution of

$$\begin{aligned} &\text{Find } \phi^k \in W_{per}(Y) \text{ such that} \\ &\int_Y \mu \frac{\partial \phi^k}{\partial y_i} \frac{\partial v}{\partial y_i} d\mathbf{y} = \int_Y \mu \frac{\partial v}{\partial y_k} d\mathbf{y}. \end{aligned} \quad (3.3.4)$$

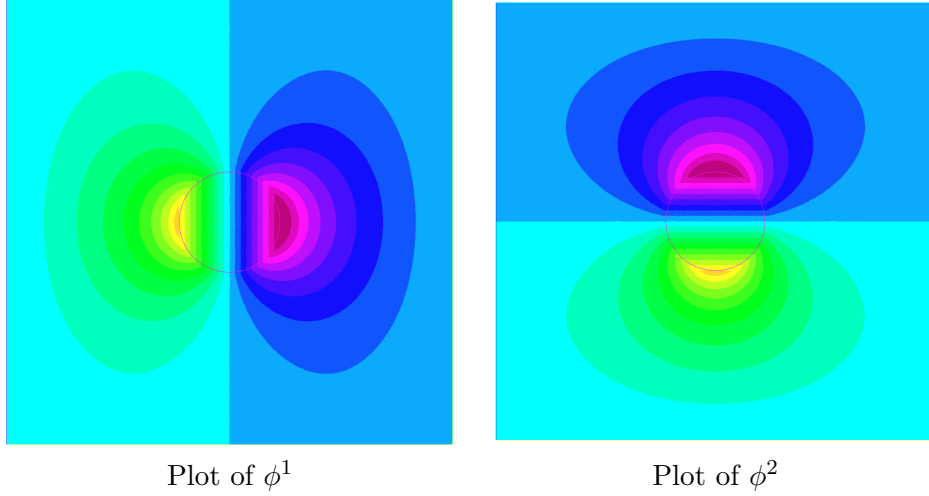


Figure 3.2: Plot of the solution ϕ^k in (3.3.4) for magnetite nanoparticles of volume fraction $\phi = 0.07$ with magnetic permeability $\mu = 8.41946 \times 10^{-6} \text{ N/A}^2$ using **FreeFem++**.

By virtue of linearity of (3.3.3) we can write

$$\psi(\mathbf{x}, \mathbf{y}) = \phi^k(\mathbf{y}) \tilde{H}_k^0(\mathbf{x}) + C(\mathbf{x}).$$

In principle, once \tilde{H}_k^0 is known, we can determine ψ up to an additive function of \mathbf{x} . Hence, combining (3.3.1) and the above relationship between ψ and ϕ^k we obtain the following constitutive law between the magnetic induction and the magnetic field,

$$\tilde{B}_i^0 = \mu_{ik} \tilde{H}_k^0, \text{ where } \mu_{ik} = \int_Y \mu \left(-\frac{\partial \phi^k}{\partial y_i} + \delta_{ik} \right) d\mathbf{y}. \quad (3.3.5)$$

One can show (see [52]) that the homogenized coefficients are symmetric, $\mu_{ik} = \mu_{ki}$. Moreover, if we denote by $A_{i\ell}(\mathbf{y}) = \left(-\frac{\partial \phi^\ell(\mathbf{y})}{\partial y_i} + \delta_{i\ell} \right)$ one can see from (3.3.4) that $H_i^0 = A_{i\ell} \tilde{H}_\ell^0$ and thus the Maxwell stress takes the following form,

$$\tau_{ij}^0 = \mu A_{i\ell} A_{jm} \tilde{H}_\ell^0 \tilde{H}_m^0 - \frac{1}{2} \mu A_{mk} A_{\ell k} \delta_{ij} \tilde{H}_m^0 \tilde{H}_\ell^0 = \mu A_{ij}^{m\ell} \tilde{H}_m^0 \tilde{H}_\ell^0.$$

Here $A_{ij}^{m\ell} = \frac{1}{2} (A_{i\ell} A_{jm} + A_{j\ell} A_{im} - A_{mk} A_{\ell k} \delta_{ij})$ and has the following symmetry, $A_{ij}^{m\ell} = A_{ji}^{m\ell} = A_{ij}^{\ell m}$. Recall that the $\text{div} \tau^\epsilon = 0$ in $\Omega_{2\epsilon}$ and $\text{div} \tau^\epsilon = -\mathbf{B}^\epsilon \times \text{curl } \mathbf{H}^\epsilon$ in $\Omega_{1\epsilon}$. From the two scale expansion, at order ϵ^{-1} from equation (3.1.3) we obtain,

$$\text{div}_y \tau^0 = 0 \text{ in } Y. \quad (3.3.6)$$

3.4 Fluid velocity and pressure

3.4.1 Study of the local problems

Problem (3.2.1)-(3.2.2), (3.2.5) is an elliptic problem in the variable $\mathbf{y} \in Y$ with forcing terms $\mathbf{v}^0(\mathbf{x})$ and $\tilde{\mathbf{H}}^0(\mathbf{x})$ at the macroscale. We can decouple the contributions of $\mathbf{v}^0(\mathbf{x})$ and $\tilde{\mathbf{H}}^0(\mathbf{x})$ and split \mathbf{v}^1 and p^0 in two parts: a part that is driven by the bulk velocity, and a part that comes from the bulk magnetic field.

$$v_k^1(\mathbf{x}, \mathbf{y}) = \chi_k^{m\ell}(\mathbf{y}) e_{m\ell}(\mathbf{v}^0) + \xi_k^{m\ell}(\mathbf{y}) \tilde{H}_m^0 \tilde{H}_\ell^0 + A_k(\mathbf{x}),$$

$$p^0(\mathbf{x}, \mathbf{y}) = p^{m\ell}(\mathbf{y}) e_{m\ell}(\mathbf{v}^0) + \pi^{m\ell}(\mathbf{y}) \tilde{H}_m^0 \tilde{H}_\ell^0 + \bar{p}^0(\mathbf{x}),$$

where $\int_{Y_f} p^{m\ell}(\mathbf{y}) d\mathbf{y} = 0$, $\int_{Y_f} \pi^{m\ell}(\mathbf{y}) d\mathbf{y} = 0$.

Here, $\chi^{m\ell}$ satisfies

$$\begin{aligned} -\frac{\partial}{\partial y_j} \varepsilon_{ij}^{m\ell} &= 0 \quad \text{in } Y_f, \\ \varepsilon_{ij}^{m\ell} &= -p^{m\ell} \delta_{ij} + 2(C_{ijm\ell} + e_{ijy}(\chi^{m\ell})) \\ -\frac{\partial \chi_i^{m\ell}}{\partial y_i} &= 0 \quad \text{in } Y_f, \\ \llbracket \chi^{m\ell} \rrbracket &= 0 \quad \text{on } S, \\ C_{ijm\ell} + e_{ijy}(\chi^{m\ell}) &= 0 \quad \text{in } T, \\ \chi^{m\ell} &\text{ is } Y\text{-periodic, } \tilde{\chi}^{m\ell} = \mathbf{0} \text{ in } Y \end{aligned} \quad (3.4.1)$$

together with the balance of forces and torques,

$$\int_S \varepsilon_{ij}^{m\ell} n_j ds = 0 \text{ and } \int_S \epsilon_{ijk} y_j \varepsilon_{kp}^{m\ell} n_p ds = 0 \quad (3.4.2)$$

The variational formulation problem of (3.4.1)-(3.4.2) is

$$\begin{aligned} &\text{Find } \chi^{m\ell} \in \mathcal{U}_{ad} \text{ such that} \\ &\int_{Y_f} 2 e_{ijy}(\chi^{m\ell}) e_{ijy}(\phi - \chi^{m\ell}) d\mathbf{y} = 0 \text{ for all } \phi \in \mathcal{U}_{ad}. \end{aligned} \quad (3.4.3)$$

where

$$\mathcal{U}_{ad} = \{\mathbf{u} \in (H_{per}^1(Y))^3 \mid \operatorname{div} \mathbf{u} = 0 \text{ in } Y_f, e_{ijy}(\mathbf{u}) = -C_{ijm\ell} \text{ in } T, \llbracket \mathbf{u} \rrbracket = \mathbf{0} \text{ on } S, \tilde{\mathbf{u}} = \mathbf{0} \text{ in } Y\}$$

is a closed, convex, non-empty subset of $(H_{per}^1(Y))^3$. Here $C_{ijm\ell} = \frac{1}{2}(\delta_{im}\delta_{j\ell} + \delta_{il}\delta_{jm}) - \frac{1}{3}\delta_{ij}\delta_{m\ell}$. We remark that if we define $B_k^{ij} = \frac{1}{2}(y_i\delta_{jk} + y_j\delta_{ik}) - \frac{1}{3}y_k\delta_{ij}$, then $e_{ij}(\mathbf{B}^{m\ell}) = C_{ijm\ell}$. Existence and uniqueness of a solution follows from classical theory of variational inequalities (see [31]).

In similar fashion we can derive the local problem for $\xi^{m\ell}$,

$$\begin{aligned} &-\frac{\partial}{\partial y_j} \Sigma_{ij}^{m\ell} = 0 \quad \text{in } Y_f, \\ &\Sigma_{ij}^{m\ell} = -\pi^{m\ell} \delta_{ij} + 2 e_{ijy}(\xi^{m\ell}) \\ &-\frac{\partial \xi_i^{m\ell}}{\partial y_i} = 0 \quad \text{in } Y_f, \\ &\llbracket \xi^{m\ell} \rrbracket = 0 \quad \text{on } S, \\ &e_{ijy}(\xi^{m\ell}) = 0 \quad \text{in } T, \\ &\xi^{m\ell} \text{ is } Y\text{-periodic, } \tilde{\xi}^{m\ell} = 0, \end{aligned} \quad (3.4.4)$$

Using (3.3.6) the balance of forces and torques reduces to

$$\int_S \Sigma_{ij}^{m\ell} n_j ds = 0 \text{ and } \int_S \epsilon_{ijk} y_j (\Sigma_{kp}^{m\ell} + \alpha \llbracket \mu A_{kp}^{m\ell} \rrbracket) n_p ds = 0. \quad (3.4.5)$$

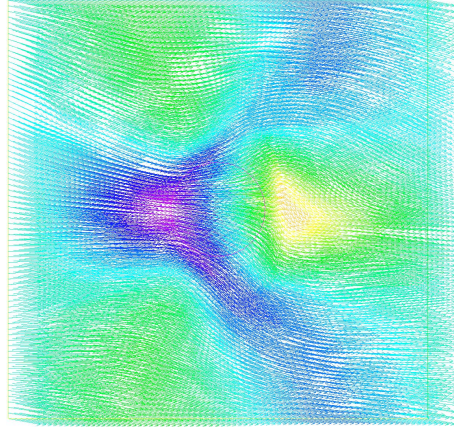
We can formulate (3.4.4)–(3.4.5) variationally as

$$\begin{aligned} &\text{Find } \xi^{m\ell} \in V_{per}(Y) \text{ such that} \\ &\int_{Y_f} 2 e_{ijy}(\xi^{m\ell}) e_{ijy}(\phi) d\mathbf{y} + \int_Y A_{ij}^{m\ell} e_{ijy}(\phi) d\mathbf{y} = 0 \text{ for all } \phi \in V_{per}(Y), \end{aligned} \quad (3.4.6)$$

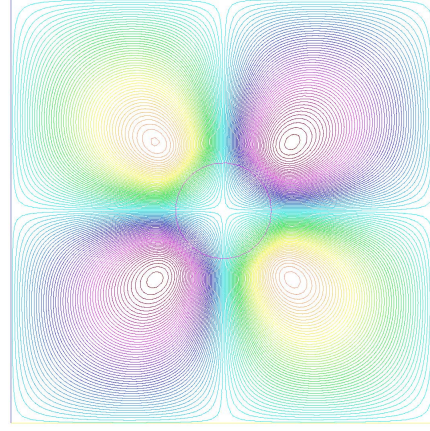
where

$$V_{per}(Y) = \{\mathbf{v} \in (H_{per}^1(Y))^3 \mid \operatorname{div} \mathbf{v} = 0 \text{ in } Y_f, e_{ijy}(\mathbf{v}) = 0 \text{ in } T, \llbracket \mathbf{v} \rrbracket = \mathbf{0} \text{ on } S, \tilde{\mathbf{v}} = \mathbf{0} \text{ in } Y\}$$

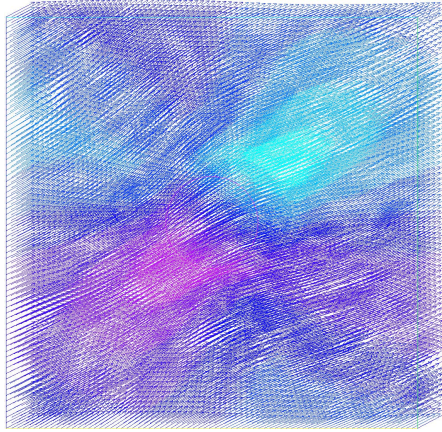
is a closed subspace of $(H_{per}^1(Y))^3$. Existence and uniqueness follows from an application of the Lax-Milgram lemma.



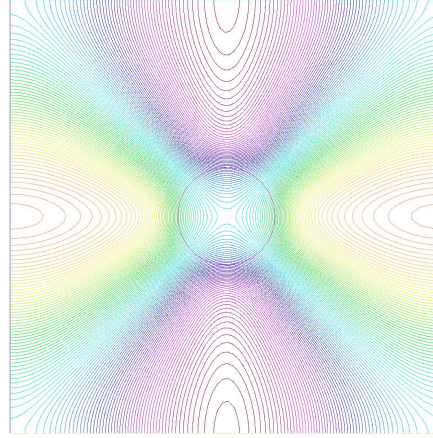
Plot of χ^{11}



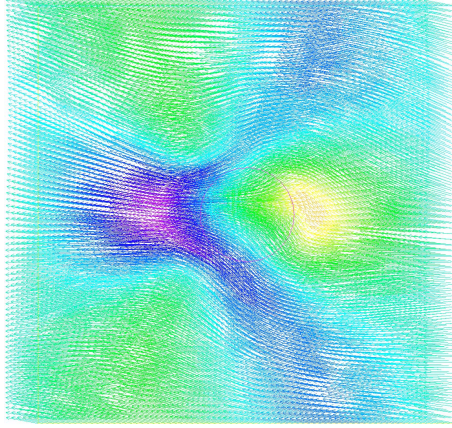
Streamline for χ^{11}



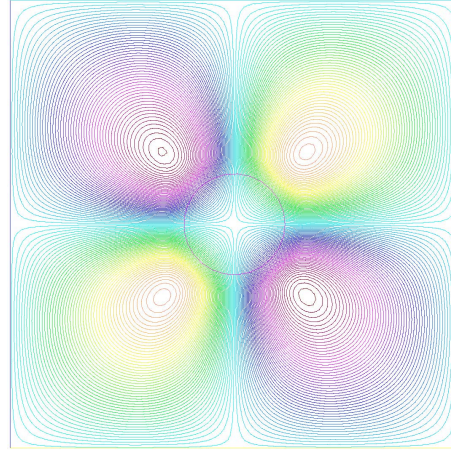
Plot of χ^{12}



Streamline of χ^{12}



Plot of χ^{22}



Streamline of χ^{22}

Figure 3.3: On the left is a plot of the solution $\chi^{m\ell}$ in (3.4.3) and on the right the corresponding streamlines for magnetite nanoparticles of volume fraction $\phi = 0.07$ generated using **FreeFem++**.

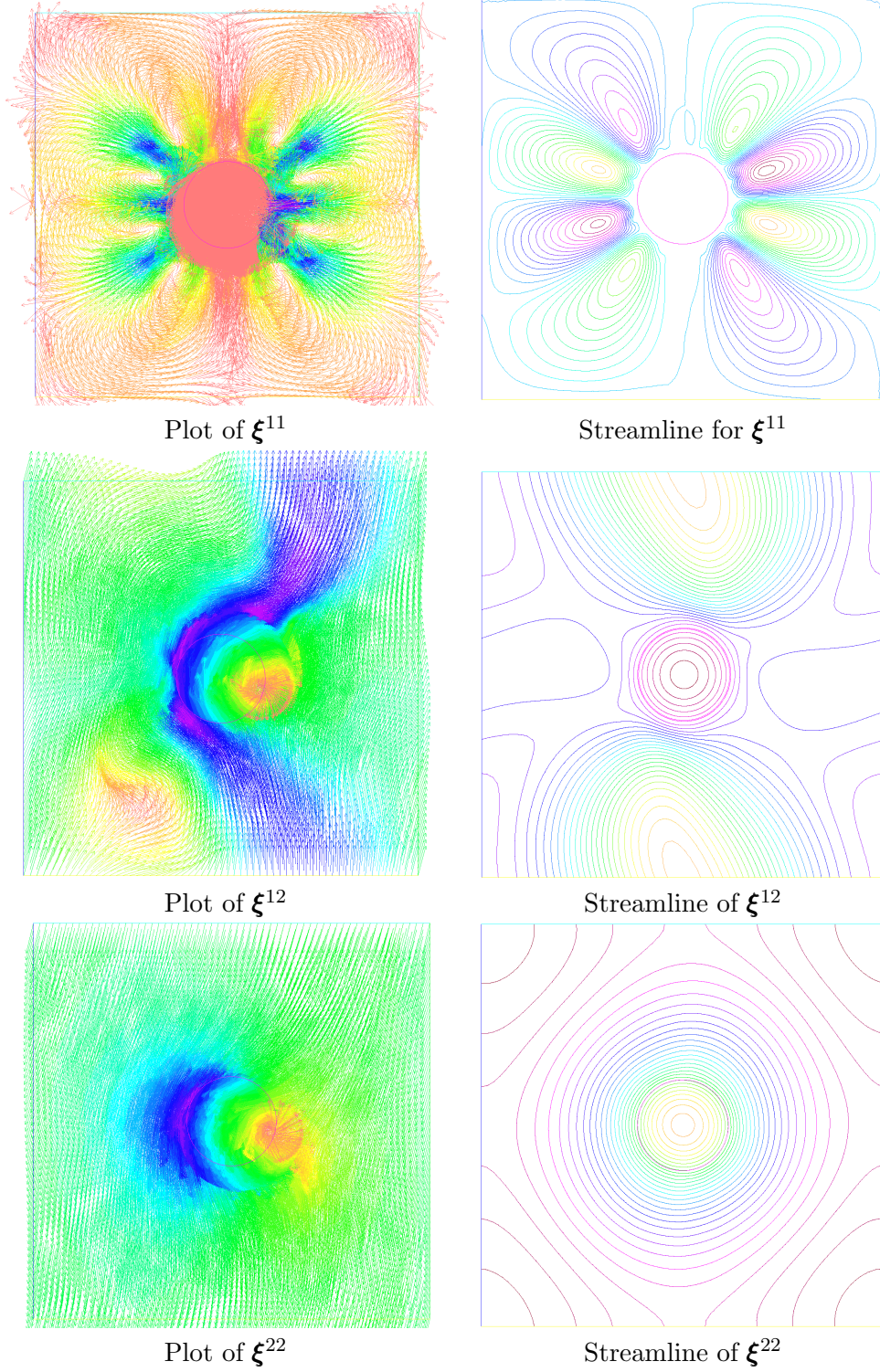


Figure 3.4: On the left is a plot of the solution $\xi^{m\ell}$ in (3.4.6) and on the right the corresponding streamlines for magnetite nanoparticles of volume fraction $\phi = 0.07$ generated using FreeFem++.

3.5 Homogenized MR fluid equations

At the ϵ^0 order we obtain the following problems,

$$-\operatorname{div}_x \sigma^0 - \operatorname{div}_y \sigma^1 = \mathbf{0} \quad \text{in } Y_f, \quad (3.5.1a)$$

$$\operatorname{div}_x \mathbf{v}^1 + \operatorname{div}_y \mathbf{v}^2 = 0 \quad \text{in } Y_f, \quad (3.5.1b)$$

$$\operatorname{div}_x \mathbf{B}^0 + \operatorname{div}_y \mathbf{B}^1 = 0 \quad \text{in } Y, \quad (3.5.1c)$$

$$\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1 = 0 \quad \text{in } Y_f, \quad (3.5.1d)$$

$$\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1 = R_m \mathbf{v}^0 \times \mathbf{B}^0 \quad \text{in } T, \quad (3.5.1e)$$

with boundary conditions

$$\begin{aligned} \llbracket \mathbf{v}^2 \rrbracket &= \mathbf{0}, \quad \llbracket \mathbf{B}^1 \cdot \mathbf{n} \rrbracket = 0 \quad \llbracket \mathbf{n} \times \mathbf{H}^1 \rrbracket = \mathbf{0} \quad \text{on } S, \\ \mathbf{v}^2, \quad \mathbf{H}^1 &\text{ are } Y\text{-periodic.} \end{aligned} \quad (3.5.2)$$

We assume a Taylor expansion both of the viscous stress and the Maxwell stress of the form (see [33]),

$$\sigma^\epsilon = \sigma^0(\mathbf{x}_C, \mathbf{y}) + \frac{\partial \sigma^0(\mathbf{x}_C, \mathbf{y})}{\partial x_\ell} (x_\ell - x_{C\ell}) + \epsilon \sigma^1(\mathbf{x}_C, \mathbf{y}) + \epsilon \frac{\partial \sigma^1(\mathbf{x}_C, \mathbf{y})}{\partial x_\ell} (x_\ell - x_{C\ell}) + \dots$$

$$\tau^\epsilon = \tau^0(\mathbf{x}_C, \mathbf{y}) + \frac{\partial \tau^0(\mathbf{x}_C, \mathbf{y})}{\partial x_\ell} (x_\ell - x_{C\ell}) + \epsilon \tau^1(\mathbf{x}_C, \mathbf{y}) + \epsilon \frac{\partial \tau^1(\mathbf{x}_C, \mathbf{y})}{\partial x_\ell} (x_\ell - x_{C\ell}) + \dots$$

where the expansion of the Maxwell stress occurs both inside the rigid particle and the fluid.

Using this method we can expand the balance of forces, (3.1.7), and obtain at order ϵ^3 ,

$$\begin{aligned} 0 &= \int_S \left(\frac{\partial \sigma_{ij}^0}{\partial x_k} y_k + \sigma_{ij}^1 \right) n_j ds + \alpha \int_S \llbracket \left(\frac{\partial \tau_{ij}^0}{\partial x_k} y_k + \tau_{ij}^1 \right) n_j \rrbracket ds \\ &\quad - \alpha \int_T (\mathbf{B}^0 \times (\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1))_i d\mathbf{y} \end{aligned} \quad (3.5.3)$$

Integrate (3.5.1a) over Y_f and add to (3.5.3) obtain the following,

$$\begin{aligned} 0 &= \int_{Y_f} \frac{\partial \sigma_{ij}^0}{\partial x_j} d\mathbf{y} + \int_S \frac{\partial \sigma_{ij}^0}{\partial x_k} y_k n_j ds + \alpha \int_S \llbracket \left(\frac{\partial \tau_{ij}^0}{\partial x_k} y_k + \tau_{ij}^1 \right) n_j \rrbracket ds \\ &\quad - \alpha \int_T (\mathbf{B}^0 \times (\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1))_i d\mathbf{y} \end{aligned} \quad (3.5.4)$$

At order ϵ^0 we obtain, $\operatorname{div}_x \tau^0 + \operatorname{div}_y \tau^1 = 0$ in Y_f and $\operatorname{div}_x \tau^0 + \operatorname{div}_y \tau^1 = -\mathbf{B}^0 \times (\operatorname{curl}_x \mathbf{H}^0 + \operatorname{curl}_y \mathbf{H}^1)$ in T . Combining the aforementioned results and the divergence theorem we can rewrite (3.5.4) the following way,

$$0 = \int_{Y_f} \frac{\partial \sigma_{ij}^0}{\partial x_j} d\mathbf{y} + \int_S \frac{\partial \sigma_{ik}^0}{\partial x_j} y_j n_k ds + \alpha \int_S \left[\frac{\partial \tau_{ik}^0}{\partial x_j} y_j n_k \right] ds + \alpha \int_Y \frac{\partial \tau_{ij}^0}{\partial x_j} d\mathbf{y}.$$

Utilizing (3.2.1b), (3.2.4) and the balance of torques of problems (3.4.1), (3.4.4) we can see that,

$$\sigma_{ik}^0 y_j + \llbracket \tau_{ik}^0 \rrbracket y_j = \frac{1}{2} (\sigma_{ik}^0 y_j + \llbracket \tau_{ik}^0 \rrbracket y_j + \sigma_{jk}^0 y_i + \llbracket \tau_{jk}^0 \rrbracket y_i).$$

Moreover, using the local problems for $\chi^{m\ell}$ and $\xi^{m\ell}$ we can write the viscous and the Maxwell stress as follows,

$$\sigma_{ij}^0 = -\bar{p}^0 \delta_{ij} + \varepsilon_{ij}^{m\ell} e_{m\ell x}(\mathbf{v}^0) + \Sigma_{ij}^{m\ell} \tilde{H}_m^0 \tilde{H}_\ell^0,$$

$$\tau_{ij}^0 = \mu A_{ij}^{m\ell} \tilde{H}_m^0 \tilde{H}_\ell^0$$

Hence, the homogenized fluid equations in Ω are,

$$\begin{aligned} 0 = & \frac{\partial}{\partial x_j} \left(-\bar{p}^0 \delta_{ij} + \left\{ \int_{Y_f} 2\nu e_{ijy}(\mathbf{B}^{m\ell} + \chi^{m\ell}) d\mathbf{y} + \int_S \varepsilon_{pk}^{m\ell} B_p^{ij} n_k ds \right\} e_{m\ell x}(\mathbf{v}^0) \right. \\ & \left. + \left\{ \int_{Y_f} 2\nu e_{ijy}(\xi^{m\ell}) d\mathbf{y} + \int_S \Sigma_{pk}^{m\ell} B_p^{ij} n_k ds + \alpha \int_Y \mu A_{ij}^{m\ell} d\mathbf{y} + \alpha \int_S \llbracket \mu A_{pk}^{m\ell} \rrbracket B_p^{ij} n_k \right\} \tilde{H}_m^0 \tilde{H}_\ell^0 \right). \end{aligned}$$

Moreover, using (3.2.1c)–(3.2.1d) and the divergence theorem we can obtain the incompressibility condition, $\operatorname{div}_x \mathbf{v}^0 = 0$.

Denote by

$$\nu_{ijm\ell} = \left\{ \int_{Y_f} 2 e_{ijy}(\mathbf{B}^{m\ell} + \chi^{m\ell}) d\mathbf{y} + \int_S \varepsilon_{pk}^{m\ell} B_p^{ij} n_k ds \right\}$$

and

$$\beta_{ijm\ell} = \left\{ \int_{Y_f} 2 e_{ijy}(\xi^{m\ell}) d\mathbf{y} + \int_S \Sigma_{pk}^{m\ell} B_p^{ij} n_k ds + \alpha \int_Y \mu A_{ij}^{m\ell} d\mathbf{y} + \alpha \int_S \llbracket \mu A_{pk}^{m\ell} \rrbracket B_p^{ij} n_k \right\}$$

Using local problem (3.4.1) we can re-write the $\nu_{ijm\ell}$ the following way,

$$\nu_{ijml} = \int_{Y_f} e_{pq}(\mathbf{B}^{ml} + \boldsymbol{\chi}^{ml}) e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y}. \quad (3.5.5)$$

In a similar fashion, using local problem (3.4.4) and the kinematic condition in (3.4.1) we can re-write β_{ijml} as follows

$$\beta_{ijml} = \int_{Y_f} 2 e_{pq}(\boldsymbol{\xi}^{ml}) e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y} + \alpha \int_{Y_f} \mu A_{pq}^{ml} e_{pq}(\mathbf{B}^{ij} + \boldsymbol{\chi}^{ij}) d\mathbf{y} + \int_Y \mu A_{ij}^{ml} d\mathbf{y}. \quad (3.5.6)$$

It is now clear that ν_{ijml} possesses the following symmetry, $\nu_{ijml} = \nu_{jiml} = \nu_{mlij}$. While for β_{ijml} , we have $\beta_{ijml} = \beta_{jiml} = \beta_{ijlm}$.

To obtain the homogenized Maxwell equations, average (3.5.1c), (3.5.1d), and (3.5.1e) over Y , Y_f , and T respectively,

$$\frac{\partial \tilde{B}_j^0}{\partial x_j} = 0, \quad \epsilon_{ijk} \frac{\partial \tilde{H}_k^0}{\partial x_j} = R_m \epsilon_{ijk} v_j^0 \mu_{kp}^S \tilde{H}_p^0 \quad \text{in } \Omega$$

where $\tilde{B}_i^0 = \mu_{ik} \tilde{H}_k^0$, $\mu_{ik} = \int_Y \mu \left(-\frac{\partial \phi^k}{\partial y_i} + \delta_{ik} \right) d\mathbf{y}$, and $\mu_{ik}^S = \int_T \mu \left(-\frac{\partial \phi^k}{\partial y_i} + \delta_{ik} \right) d\mathbf{y}$ as in (3.3.5), and with the following boundary conditions,

$$\tilde{H}_i^0 = b_i, \quad v_i^0 = 0 \quad \text{on } \partial\Omega$$

The effective coefficients are computed as the angular averaging of the tensors ν_{ijml} and β_{ijml} . This is done by introducing the projection on hydrostatic fields, P_b , and the projection on shear fields P_s (see [38]). The components of the projections in three dimensional space are given by:

$$(P_b)_{ijkl} = \frac{1}{3} \delta_{ij} \delta_{kl}, \quad (P_s)_{ijkl} = \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{3} \delta_{ij} \delta_{kl}$$

Let us make the following notations:

$$\begin{aligned} \nu_b &= \text{tr}(P_b \nu) = \frac{1}{3} \nu_{ppqq}, & \nu_s &= \text{tr}(P_s \nu) = \left(\nu_{pqpq} - \frac{1}{3} \nu_{ppqq} \right), \\ \beta_b &= \text{tr}(P_b \beta) = \frac{1}{3} \beta_{ppqq}, & \beta_s &= \text{tr}(P_s \beta) = \left(\beta_{pqpq} - \frac{1}{3} \beta_{ppqq} \right). \end{aligned}$$

Then we can re-write the homogenized coefficients ν_{ijml} and β_{ijml} as follows:

$$\begin{aligned} \nu_{ijml} &= \frac{1}{3} (\nu_b - \nu_s) \delta_{ij} \delta_{ml} + \frac{1}{2} \nu_s (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \\ \beta_{ijml} &= \frac{1}{3} (\beta_b - \beta_s) \delta_{ij} \delta_{ml} + \frac{1}{2} \beta_s (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}). \end{aligned}$$

Gathering all the equations we have that the homogenized equations governing the MR fluid form the following coupled system between the Stokes equations and the quasistatic Maxwell equations,

$$\begin{aligned}
\frac{\partial}{\partial x_j} (\sigma_{ij}^H + \tau_{ij}^H) &= 0, \quad \frac{\partial v_i^0}{\partial x_i} = 0 && \text{in } \Omega, \\
\sigma_{ij}^H + \tau_{ij}^H &= -\bar{p}^0 \delta_{ij} + \nu_s e_{ij}(\mathbf{v}^0) + \frac{1}{3} (\beta_b - \beta_s) \delta_{ij} \left| \tilde{\mathbf{H}}^0 \right|^2 + \beta_s \tilde{H}_i^0 \tilde{H}_j^0 && (3.5.7) \\
\frac{\partial(\mu_{jk} \tilde{H}_k^0)}{\partial x_j} &= 0, \quad \epsilon_{ijk} \frac{\partial \tilde{H}_k^0}{\partial x_j} = R_m \epsilon_{ijk} v_j^0 \mu_{kp}^S \tilde{H}_p^0 && \text{in } \Omega, \\
v_i^0 &= 0, \quad \tilde{H}_i^0 = b_i && \text{in } \Omega.
\end{aligned}$$

In the next section we will consider the flow of the MR fluid flow between infinite planar plates and compare its profile velocity to both Poiseuille and Couette flow.

3.6 Velocity profile of the MR fluid

In this section we compute the cross sectional velocity profiles of Poiseuille and Couette flow for spherical suspensions of rigid particles. We denote by $\mathbf{v} = (v_1, v_2)$ the two dimensional velocity and by $\mathbf{H} = (H_1, H_2)$ the two dimensional magnetic field. Thus, the two dimensional MR equations in (3.5.7) reduce to the following,

$$\nu \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} \right) - \frac{\partial \pi^0}{\partial x_1} + \frac{\partial}{\partial x_1} \left(\frac{1}{2} \beta (H_1^2 - H_2^2) \right) + \frac{\partial}{\partial x_2} (\beta H_1 H_2) = 0 \quad (3.6.1a)$$

$$\nu \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} \right) - \frac{\partial \pi^0}{\partial x_2} + \frac{\partial}{\partial x_1} (\beta H_1 H_2) + \frac{\partial}{\partial x_2} \left(\frac{1}{2} \beta (H_2^2 - H_1^2) \right) = 0 \quad (3.6.1b)$$

$$\frac{\partial}{\partial x_1} (\mu H_1) + \frac{\partial}{\partial x_2} (\mu H_2) = 0 \quad (3.6.1c)$$

$$\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial x_2} = \eta \mu^S (v_1 H_2 - v_2 H_1) \quad (3.6.1d)$$

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} = 0 \quad (3.6.1e)$$

3.6.1 Poiseuille flow

We consider the problem of a unidirectional steady flow between two infinite, parallel, stationary plates that are non-conducting and non-magnetizable aligned along the x_1 -axis of distance ℓ apart (see FIGURE 3.5). We apply a stationary magnetic field \mathbf{H} on the bottom plate. Since we are dealing with infinite plates, the velocity \mathbf{v} depends only on x_2 . Using (3.6.1e) we immediately obtain that v_2 is constant and since the plates are stationary $v_2 = 0$.



Figure 3.5: Stationary plates of distance ℓ apart.

Since the flow is unidirectional, we expect that the component of the magnetic field perpendicular to the flow to be constant, $H_2 = K$, while the component parallel to the flow to depend on the fluid velocity. Hence, using (3.6.1c) we obtain the $H_1(x_1, x_2) = H_1(x_2)$. Therefore, the equations in (3.6.1) reduce to the following,

$$\nu \frac{\partial^2 v_1}{\partial x_1^2} + \beta K \frac{\partial H_1}{\partial x_2} = \frac{\partial \pi^0}{\partial x_1} \quad (3.6.2a)$$

$$-\frac{\partial \pi^0}{\partial x_2} - \frac{1}{2} \beta \frac{\partial H_1^2}{\partial x_2} = 0 \quad (3.6.2b)$$

$$-\frac{\partial H_1}{\partial x_2} = \eta \mu^S K v_1 \quad (3.6.2c)$$

Making use of (3.6.2b) we obtain that $\pi^0(x_1, x_2) + \frac{1}{2} \beta H_1(x_2)^2$ is a function of only x_1 and therefore by differentiating the expression with respect to x_1 we get that $\frac{\partial \pi^0}{\partial x_1}$ is a function only x_1 . Therefore, on (3.6.2a) the left hand side is a function of x_2 and the right hand side is a function of x_1 . Thus they have to be constant. Substituting (3.6.2c) in (3.6.2a) we obtain the following differential equations,

$$\frac{d^2 v_1}{d x_2^2} - \lambda^2 v_1 = C_p \quad (3.6.3a)$$

$$\frac{\partial \pi^0}{\partial x_1} = C_p \quad (3.6.3b)$$

Here $\lambda = \sqrt{\frac{\eta \mu^S \beta}{\nu}} K$.

The general solution of (3.6.3a) is

$$v_1(x_2) = c_1 e^{\lambda x_2} + c_2 e^{-\lambda x_2} + \frac{C_p}{\nu \lambda^2}$$

Given that $v_1(0) = v_1(1) = 0$ we have,

$$v_1(x_2) = \frac{C_p}{\nu \lambda^2} \left(\frac{\sinh(\lambda x_2) - \sinh(\lambda (x_2 - 1))}{\sinh(\lambda)} - 1 \right)$$

Once the velocity $v_1(x_2)$ is known, we can use (3.6.2c) to compute $H_1(x_2)$ with boundary condition $H_1(0) = K_1$ and obtain,

$$H_1(x_2) = \eta \mu^s K \frac{C_p}{\nu \lambda^3 \sinh(\lambda)} (-\cosh(\lambda x_2) + \cosh(\lambda(x_2 - 1)) - \cosh(\lambda) + 1) + K_1.$$

We remark that as K tends to zero we have

$$\lim_{K \rightarrow 0} v_1(x_2) = \frac{C_p}{2\nu} x_2 (x_2 - 1)$$

which is precisely the profile of Poiseuille flow with stationary plates at $x_2 = 0$ and $x_2 = 1$.

3.6.2 Couette flow

The setting and calculations for the unidirectional Couette flow are the same as Poiseuille flow. In a similar way, we can carry out computations for the plane Couette flow. For simplicity we assumed the bottom plate is the x_1 axis and the top plate is at $x_2 = 1$ (see FIGURE 3.6). A shear stress, $\dot{\gamma}$, is applied to the top plate and a no-slip condition is applied to the bottom plate. Thus, we solve (3.6.3a) with initial conditions $v_1(0) = 0$ and $v_1'(1) = \dot{\gamma}$ and obtain

$$v_1(x_2) = \frac{\dot{\gamma} \nu \lambda \sinh(\lambda x_2) + C_p \cosh(\lambda(x_2 - 1))}{\nu \lambda^2 \cosh(\lambda)} - \frac{C_p}{\nu \lambda^2}$$

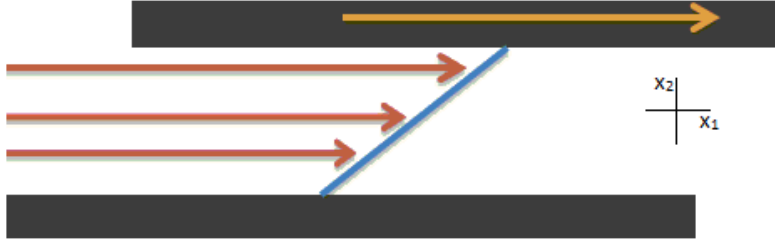


Figure 3.6: Fixed bottom plate with an applied strain of $\dot{\gamma}$ on the top plate.

Again, we remark that

$$\lim_{K \rightarrow 0} v_1(x_2) = \frac{C_p}{2\nu} x_2 (x_2 - 2) + \dot{\gamma} x_2$$

which is precisely Couette flow with forces applied on the plate lying in $x_2 = 1$.

To compute H_1 we use (3.6.2c) to obtain

$$H_1(x_2) = \frac{\lambda \dot{\gamma} \nu}{\nu \lambda^3 \cosh(\lambda)} (\cosh(\lambda x_2) - 1) + \frac{C_p}{\nu \lambda^3 \cosh(\lambda)} (\sinh(\lambda(x_2 - 1)) - \sinh(\lambda)) - \frac{C_p x_2}{\nu \lambda^2} + K_1.$$

3.6.3 Magnetite nanoparticles

In this section we consider a suspension of spherical magnetite nanoparticles in de-ionized water of viscosity 0.001 Pa with volume fraction $\phi = 0.07$. The electrical conductivity of the nanoparticles is assumed to be 20,000 S/m, while the magnetic permeability is $8.41946 \times 10^{-6} \text{ N/A}^2$ for the nanoparticles and $1.25662 \times 10^{-6} \text{ N/A}^2$ for the water. Carrying out explicit computations of the effective coefficients we can plot the profiles of the MR flow, for different values of the magnetic field and compare them with the Poiseuille flow profile (see FIGURE 3.7).

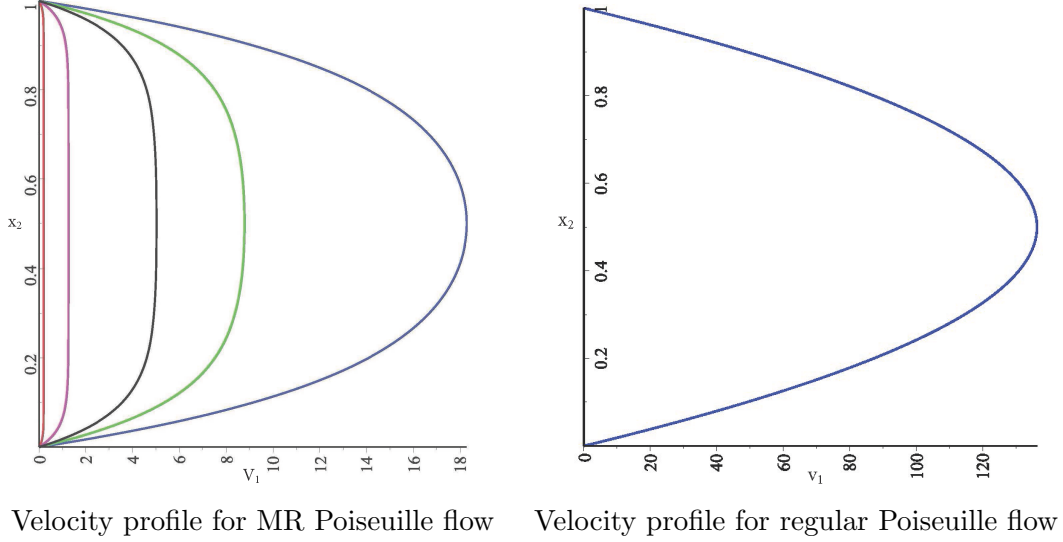


Figure 3.7: The plots on the left represent the velocity profile for $B_2=0.05, 0.02, 0.01, 0.0075, 0.005 \text{ T}$ (from left to right). The plot on the right is the velocity profile $B_2=0 \text{ T}$.

For shear experiments, the response of of MR fluids is often modeled using a Bingham constitutive law [47], [20], [21]. Although the Bingham constitutive law measures the response of the MR fluid quite reasonably, actual MR fluid behavior exhibits departures from the Bingham model [61],[22]. From the plot above we can see that for low values of the magnetic field the flow does not resemble a Bingham flow. However, as the magnetic field intensity increases the flow begins to approach a Bingham fluid flow.

Similarly, we can plot and compare the velocity profile of MR Couette flow against regular Couette flow for zero pressure gradient,

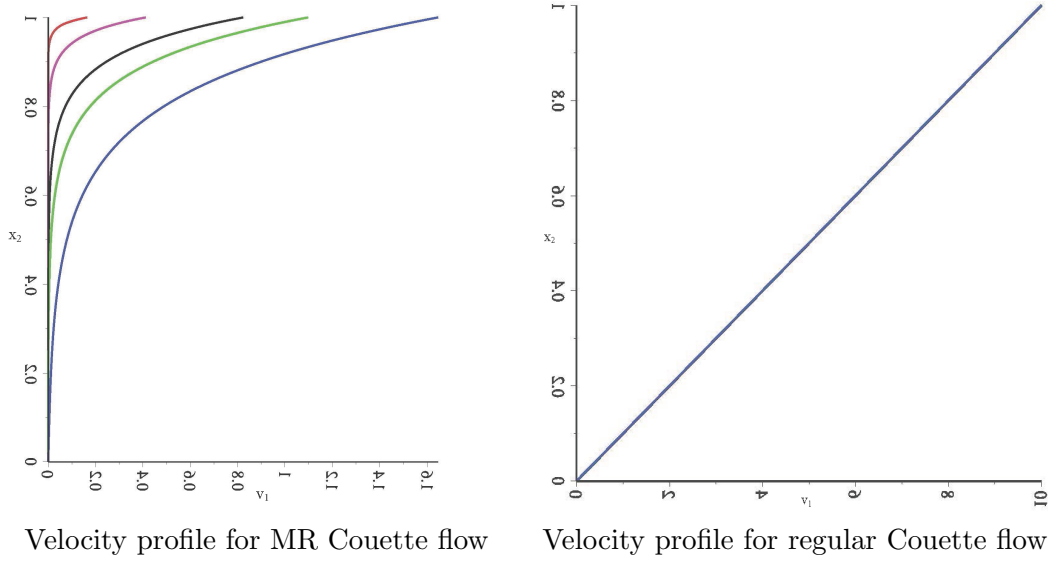


Figure 3.8: The plots on the left represent the velocity profile for $B_2=0.05, 0.02, 0.01, 0.0075, 0.005$ T (left to right). On the right the velocity profile for $B=0$ T.

The formation of the chain like structures mentioned in the introduction play a critical role in controlling the rheological properties of the MR fluid. When a steady shear is applied the chain like structures resist against the deformation up to a critical point. This is what causes the fluid to exhibit a yield stress and the Bingham constitutive may be applicable. The plot in FIGURE 3.9 depicts the stress vs shear rate curve relationship. When $K_1 = 0$ there is no yield stress present. However, for very small non-zero values of K_1 we obtain the results of [61] for the linear portion of the stress vs shear rate curve at high shear rates. Additionally, we are able to match their extrapolated Bingham yield stress values.

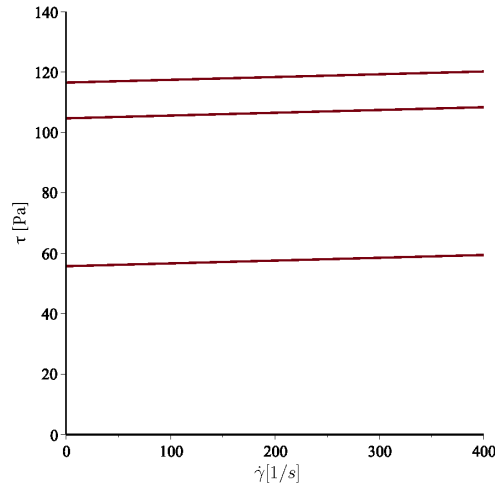


Figure 3.9: The stress versus the shear rate curve for three different magnetic fields, $B_2 = 0.288, 0.230, 0.058$.

3.7 Conclusions

We consider a suspension of rigid magnetizable particles in an non-magnetizable, non-conducting aqueous viscous fluid. In (3.3.4), (3.4.3), (3.4.6) we derive the local problems that arise from the Maxwell equations, the bulk velocity and the bulk magnetic field and obtain new constitutive laws. The effective equations governing the behavior of the MR fluid are presented in (3.5.7). The proposed model generalizes the one in [46] by coupling the velocity field with the magnetic field intensity and the effective coefficients computed generalize those in [27]. Unidirectional velocity profiles of Poiseuille and Couette flows are computed for magnetite nanoparticles of volume fraction $\phi = 0.07$ to ascertain experimental data for the stress-strain relationship of MR flows.

Appendix A

A.1 Equivalence between PDE and variational formulation for the emulsion problem

In this section of the appendix we detail the variational form used in Chapters one and two. We seek a vector function \mathbf{u}^ϵ representing the velocity of the fluid, and a scalar function p^ϵ representing the pressure, which are defined in $\Omega_{1\epsilon} \cup \Omega_{2\epsilon}$ and satisfy the following equations and boundary conditions:

$$-\operatorname{div} \sigma^\epsilon = \mathbf{f} \text{ in } \Omega_{1\epsilon} \cup \Omega_{2\epsilon}, \quad (\text{A.1.1a})$$

$$\sigma^\epsilon = -2\mu^\epsilon e(\mathbf{v}^\epsilon) + p^\epsilon I, \quad (\text{A.1.1b})$$

$$\operatorname{div} \mathbf{v}^\epsilon = 0 \text{ in } \Omega, \quad (\text{A.1.1c})$$

with boundary conditions on the surface of each droplet T_ℓ^ϵ , $\ell \in N^\epsilon$

$$[\![\mathbf{v}^\epsilon]\!] = \mathbf{0} \quad \text{on } S_\ell^\epsilon, \quad (\text{A.1.2a})$$

$$\mathbf{v}^\epsilon = \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \quad \text{on } S_\ell^\epsilon, \quad (\text{A.1.2b})$$

$$\mathbf{v}^\epsilon = \mathbf{0} \quad \text{on } \Gamma, \quad (\text{A.1.2c})$$

and an additional condition that comes from the balance of torques

$$\int_{S_\ell^\epsilon} (\mathbf{x} - \mathbf{x}_C^\ell) \times [\![\sigma^\epsilon \mathbf{n}]\!] ds = 0. \quad (\text{A.1.3})$$

If \mathbf{f} , \mathbf{u}^ϵ , and p^ϵ are smooth function satisfying (A.1.1) – (A.1.3) then, taking the scalar product of (A.1.1a) with a function \mathbf{w} in \mathcal{V}^ϵ , where

$$\mathcal{V}^\epsilon = \left\{ \mathbf{w} \in C_0^\infty(\Omega)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega, \quad \mathbf{w} = \boldsymbol{\omega}^{\ell,\epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell) \text{ on } S_\ell^\epsilon, \quad \mathbf{c} \in \mathbb{R}^3 \right\},$$

we obtain:

$$-\int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \operatorname{div} \sigma^\epsilon \cdot \mathbf{w} d\mathbf{x} = \int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \mathbf{f} \cdot \mathbf{w} d\mathbf{x}.$$

Thus we get,

$$-\sum_{\ell \in N^\epsilon} \int_{S_\ell^\epsilon} \llbracket \sigma^\epsilon \mathbf{n} \rrbracket \cdot \mathbf{w} \, d\mathbf{x} + \int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \sigma^\epsilon : \nabla \mathbf{w} \, d\mathbf{x} = \int_{\Omega_{1\epsilon} \cup \Omega_{2\epsilon}} \mathbf{f} \cdot \mathbf{w} \, d\mathbf{x},$$

and using condition (A.1.3), properties of symmetric matrices, and the fact that the fluid is incompressible we get:

$$\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, d\mathbf{x}, \text{ for all } \mathbf{w} \in \mathcal{V}^\epsilon. \quad (\text{A.1.4})$$

Equality (A.1.4) is still valid by continuity for each $\mathbf{w} \in V^\epsilon$, the closure of \mathcal{V}^ϵ in $H_0^1(\Omega)^3$. Therefore we have the following conclusion:

$$\mathbf{v}^\epsilon \in V^\epsilon \text{ and satisfies } \int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\mathbf{w}) \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{w} \, d\mathbf{x} \text{ for all } \mathbf{w} \in V^\epsilon. \quad (\text{A.1.5})$$

Conversely, assume that $\mathbf{v}^\epsilon \in V^\epsilon$ satisfies (A.1.5). Since, $\mathbf{v}^\epsilon \in H_0^1(\Omega)^3$ we immediately get $\mathbf{v}^\epsilon = \mathbf{0}$ on Γ in the sense of the traces. Furthermore, since $\mathbf{v}^\epsilon \in V^\epsilon$ we obtain that $\text{div} \mathbf{v}^\epsilon = 0$ in the distributional sense, $\llbracket \mathbf{v}^\epsilon \rrbracket = \mathbf{0}$ and $\mathbf{v}^\epsilon = \boldsymbol{\omega}^{\ell, \epsilon} \times (\mathbf{x} - \mathbf{x}_C^\ell)$ on S_ℓ^ϵ , for $\boldsymbol{\omega}^{\ell, \epsilon} \in \mathbb{R}^3$, in the sense of the traces.

Let $\boldsymbol{\phi} \in \mathcal{D}(\Omega_{1\epsilon})$ with $\text{div} \boldsymbol{\phi} = 0$ then $\boldsymbol{\phi} \in \mathcal{V}^\epsilon$ (pick $\boldsymbol{\omega}^{\ell, \epsilon} \in \mathbb{R}^3$ to be the zero vector) and using (A.1.5), we get

$$\langle -\text{div}(2\mu_1 e(\mathbf{v}^{1\epsilon})) - \mathbf{f}, \boldsymbol{\phi} \rangle = 0 \text{ for all } \boldsymbol{\phi} \in \mathcal{D}(\Omega_{1\epsilon}).$$

Using Propositions 1.1 and 1.2 in [58] there exists a distribution $p_1^\epsilon \in \mathbf{L}^2(\Omega_{1\epsilon})$ such that

$$-\text{div}(2\mu_1 e(\mathbf{v}^{1\epsilon})) - \mathbf{f} = -\nabla p_1^\epsilon,$$

in the sense of distributions in $\Omega_{1\epsilon}$. Similarly if we pick a $\boldsymbol{\phi} \in \mathcal{D}(\Omega_{2\epsilon})$ with $\text{div} \boldsymbol{\phi} = 0$ and proceed the same way as above we will obtain,

$$-\text{div}(2\mu_2 e(\mathbf{v}^{2\epsilon})) - \mathbf{f} = -\nabla p_2^\epsilon,$$

in the sense of distributions in $\Omega_{2\epsilon}$.

The last condition that we need to recover is (A.1.3). Consider $\boldsymbol{\phi} \in \mathcal{V}^\epsilon$, using (A.1.5)

$$\int_{\Omega} 2\mu^\epsilon e(\mathbf{v}^\epsilon) : e(\boldsymbol{\phi}) \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \boldsymbol{\phi} \, d\mathbf{x} \text{ for all } \boldsymbol{\phi} \in \mathcal{V}^\epsilon,$$

adding the pressure distribution in the above equation and integrating by parts we obtain:

$$\sum_{\ell \in N^\epsilon} \int_{S_\ell^\epsilon} \llbracket \sigma^\epsilon \mathbf{n} \rrbracket \cdot \boldsymbol{\phi} \, ds + \langle -\text{div} \sigma^\epsilon - \mathbf{f}, \boldsymbol{\phi} \rangle = 0,$$

which implies condition (A.1.3).

A.2 Weak formulation, uniqueness and existence of the local problem

In this section of the appendix we provide the proofs for the results used in section 1.5. We begin first with a helpful lemma. Define B_R to be the ball of radius R centered at zero. Let T compact be subset of $Y = (-\frac{1}{2}, \frac{1}{2})^3$ with Lipschitz boundary and for any $R > R_0$, assume that $Y \subset\subset B_{R_0}$.

Introduce the following space,

$$V = \{\mathbf{v} \in H_0^1(B_{R_0} \setminus Y)^3 \mid \operatorname{div} \mathbf{v} = 0\}$$

Since V is a closed subspace of $H_0^1(B_{R_0} \setminus Y)^3$ we have the decomposition

$$H_0^1(B_{R_0} \setminus Y)^3 = V \oplus V^\perp$$

PROPOSITION A.2.1. *There exists a constant C such that $\|q\|_{L_0^2(B_{R_0})} \leq \|\nabla q\|_{H^{-1}(B_{R_0} \setminus Y)^3}$ for every $q \in L_0^2(B_{R_0})$. Furthermore, there exists a constant C , independent of R , such that if $\phi \in L_0^2(B_R)$ then there exists $\mathbf{u} \in H_0^1(B_R \setminus Y)^3$ with $-\operatorname{div} \mathbf{u} = \phi$ and $\|\nabla \mathbf{u}\|_{L^2(B_R \setminus Y)^3} \leq C \|\phi\|_{L_0^2(B_R)}$ for every $R > R_0$.*

Proof. For the first part the proof relies on the closed range theorem of Banach. First note that

$$-\operatorname{grad} \in \mathcal{L}(L_0^2(B_{R_0}), H^{-1}(B_{R_0} \setminus Y)^3)$$

is the dual operator of

$$\operatorname{div} \in \mathcal{L}(H_0^1(B_{R_0} \setminus Y)^3, L^2(B_{R_0})).$$

Let $p_n \in L^2(B_{R_0})$ such that $-\operatorname{grad} p_n \rightarrow \mathbf{f}$ in $H^{-1}(B_{R_0} \setminus Y)^3$. Then for all $\mathbf{w} \in V$

$$\langle \mathbf{f}, \mathbf{w} \rangle_{H^{-1}(B_{R_0} \setminus Y)^3, H_0^1(B_{R_0} \setminus Y)^3} = 0$$

By the closed range theorem of Banach

$$\mathcal{R}(\operatorname{grad}) = N(\operatorname{div})^\perp$$

where $N(\operatorname{div})^\perp = \{\mathbf{y} \in H^{-1}(B_{R_0} \setminus Y)^3 \mid \langle \mathbf{y}, \mathbf{v} \rangle_{H^{-1}(B_{R_0} \setminus Y)^3, H_0^1(B_{R_0} \setminus Y)^3} = 0 \text{ for all } \mathbf{v} \in V\}$.

Hence, $\mathbf{f} = \nabla p$ for some $p \in L_0^2(B_{R_0})$ and by PROPOSITION 1.2(ii) in Temam [58] the estimate follows.

For the second part, we note that the range space of div is a closed subspace of $L_0^2(B_{R_0})$. By the open mapping theorem and a change of variables of the form $\mathbf{y} = \frac{R}{R_0} \mathbf{x}$ the second estimate follows. \square

We now show the existence and uniqueness of a solution to the local problem (2.4.1). The setting is as before, where $T \subset\subset B_R$ with smooth enough boundary $S = \partial T$. In such a setting we consider the following problem

$$\begin{aligned}
-\operatorname{div} \sigma^{kR} &= \mathbf{0} && \text{in } B_R \setminus S \\
\sigma_{ij}^{kR} &= -s^{kR} \delta_{ij} + 2\mu e_{ij}(\chi^{kR}) \\
\operatorname{div} \chi^{kR} &= 0 && \text{in } B_R \\
\llbracket \chi^{kR} \rrbracket &= \mathbf{0} && \text{on } S \\
\chi^{kR} &= -\mathbf{e}_k + \boldsymbol{\omega}^{kR} \times \mathbf{y} && \text{on } S \\
\chi^{kR} &= \mathbf{0} && \text{on } \partial B_R
\end{aligned} \tag{A.2.1}$$

together with the balance of torques,

$$\int_S \mathbf{y} \times \llbracket \sigma^{kR} \mathbf{n} \rrbracket ds = 0 \tag{A.2.2}$$

Define

$\mathcal{U}_{ad} = \{\mathbf{w} \in H_0^1(B_R)^3 \mid \operatorname{div} \mathbf{w} = 0 \text{ on } B_R, \quad \mathbf{w} = -\mathbf{e}_k + \boldsymbol{\omega}^{kR} \times \mathbf{y} \text{ on } S\}$
a closed convex subset of $H_0^1(B_R)^3$. Moreover, define the symmetric, bilinear form

$$\begin{aligned}
a : H_0^1(B_R)^3 \times H_0^1(B_R)^3 &\mapsto \mathbb{R} \\
a(\mathbf{u}, \mathbf{v}) &= \int_{B_R} 2\mu e_{ij}(\mathbf{u}) e_{ij}(\mathbf{v}) d\mathbf{y}
\end{aligned}$$

The weak formulation problem of (A.2.1) is

$$\begin{aligned}
&\text{Find } \chi^{kR} \in \mathcal{U}_{ad} \text{ such that} \\
&a(\chi^{kR}, \boldsymbol{\phi} - \chi^{kR}) = 0 \text{ for all } \boldsymbol{\phi} \in \mathcal{U}_{ad}.
\end{aligned} \tag{A.2.3}$$

LEMMA A.2.2. *The set \mathcal{U}_{ad} is closed, convex and non-empty subset of $H_0^1(B_R)^3$.*

Proof. Take $\mathbf{v} \in H_0^1(B_R)^3$ with $\mathbf{v} = -\mathbf{e}_i + \boldsymbol{\omega}^{kR} \times \mathbf{y}$ on S . We note that since $\operatorname{div} \mathbf{v}$ has mean zero by PROPOSITION A.2.1, there exists $\mathbf{u} \in H_0^1(B_R \setminus Y)^3$ with $\operatorname{div} \mathbf{u} = \operatorname{div} \mathbf{v}$. From this we get that $\mathbf{v} - \mathbf{u} \in H_0^1(B_R)^3$, $\operatorname{div}(\mathbf{v} - \mathbf{u}) = 0$, and $\mathbf{v} - \mathbf{u} = -\mathbf{e}_i + \boldsymbol{\omega}^{kR} \times \mathbf{y}$ on S . Hence, \mathcal{U}_{ad} is non-empty.

To show that \mathcal{U}_{ad} is closed, consider, $\mathbf{w}^n \in \mathcal{U}_{ad}$ such that $\mathbf{w}^n \rightarrow \mathbf{w}$ in $H_0^1(B_R)^3$ then

$$\begin{aligned}
\int_{\partial T} |\mathbf{w}^n|^2 ds &= \int_{\partial T} w_j^n w_j^n ds + \int_{\partial T} \epsilon_{jpq} \epsilon_{jkl} \omega_p^{kRn} c_k^n x_q x_l ds \\
&= \int_{\partial T} (\delta_{pk} \delta_{ql} - \delta_{pl} \delta_{qk}) \omega_p^{kRn} c_k^n x_q x_l ds \\
&= \int_{\partial T} \{|\boldsymbol{\omega}^{kRn}|^2 |\mathbf{x}|^2 - |\boldsymbol{\omega}^{kRn} \cdot \mathbf{x}|\} ds
\end{aligned}$$

Since $\mathbf{w}^n \rightarrow \mathbf{w}$ in $H_0^1(B_R)^3$ then $|\int_{\partial T} |\mathbf{w}^n|^2 ds| \leq C$ where C is a constant. Thus, we get that $|\mathbf{c}^n| \leq C$ and therefore each component $|c_j^n| \leq C$, for $j = 1, 2, 3$. Thus, there exists a convergent subsequence, $\omega_j^{kR^{n_k}} \rightarrow \omega_j^{kR}$ in \mathbb{R} . Hence, $\mathbf{w} = \boldsymbol{\omega}^{kR} \times \mathbf{y}$ and the closeness of \mathcal{U}_{ad} follows. The fact that \mathcal{U}_{ad} is convex follows directly from the definition. \square

THEOREM A.2.3. *The weak solution of problem (A.2.3) exists and is unique .*

Proof. Using a combination of Korn's first inequality and the Poincare inequality one can show that the bilinear form is coersive over $H_0^1(B)^3$ i.e. $a(\mathbf{u}, \mathbf{u}) \geq C \|\mathbf{u}\|_{H_0^1(B)^3}$. Moreover, since

$$\mathbf{v} \mapsto \{\|\mathbf{v}\|_{L^2(B_R)^3} + \|e(\mathbf{v})\|_{L^2(B_R)^{3 \times 3}}\}^{1/2}$$

is norm equivalent to the norm $\|\cdot\|_{H_0^1(B_R)^3}$, we have $a(\cdot, \cdot) : H_0^1(B_R)^3 \times H_0^1(B_R)^3 \mapsto \mathbb{R}$ is continuous. Hence, by THEOREM 2.1 in [31] we obtain the desired result. \square

Alternatively, since the bilinear form $a(\mathbf{u}, \mathbf{v})$ is symmetric, we can classify $\boldsymbol{\chi}^{kR}$, as the unique solution of

$$\begin{aligned} &\text{Find } \boldsymbol{\chi}^{kR} \in H_0^1(B_R)^3 \text{ such that,} \\ &\mathfrak{S}_R(\boldsymbol{\chi}^{kR}) = \min_{\mathbf{u} \in H_0^1(B_R)^3} \mathfrak{S}_R(\mathbf{u}), \end{aligned} \tag{A.2.4}$$

where

$$\mathfrak{S}_R(\mathbf{u}) = a(\mathbf{u}, \mathbf{u}) + I_{\mathcal{U}_{ad}}(\mathbf{u}), \tag{A.2.5}$$

Remark A.2.4. Assume that R_1 and R_2 are two real positive numbers such that $R_2 > R_1$. Consider two solutions $\boldsymbol{\chi}^{kR_1}$ and $\boldsymbol{\chi}^{kR_2}$ of (A.2.3). Using classification (A.2.4), each solution is a minimum over its respective domain. Hence,

$$\mathfrak{S}_{R_2}(\boldsymbol{\chi}^{kR_2}) \leq \mathfrak{S}_{R_2}(\boldsymbol{\chi}^{kR_1}) = \mathfrak{S}_{R_1}(\boldsymbol{\chi}^{kR_1}).$$

A.3 Bounds on the stresses

In this section of the appendix we obtain the bounds on the stresses needed in section 1.6. This provides the details behind the result used in [37] in the context of stochastic homogenization of a fluid flow through a membrane.

Consider the solutions $\{\boldsymbol{\chi}^{iR}, \eta^{iR}\}$ and $\{\boldsymbol{\chi}^i, \eta^i\}$ of the following auxiliary local problem in B_R and \mathbb{R}^3 respectively

$$\left\{ \begin{array}{ll} -\mu(\mathbf{y}) \Delta \boldsymbol{\chi}^{iR} + \nabla \eta^{iR} = \mathbf{0} & \text{in } B_R \setminus S, \\ \operatorname{div} \boldsymbol{\chi}^{iR} = 0 & \text{in } B_R, \\ \llbracket \boldsymbol{\chi}^{iR} \rrbracket = \mathbf{0} & \text{on } S, \\ \boldsymbol{\chi}^{iR} = -\mathbf{e}_i + \boldsymbol{\omega}^{iR} \times \mathbf{y} & \text{on } S, \\ \boldsymbol{\chi}^{iR} = \mathbf{0} & \text{on } \partial B_R, \end{array} \right. \quad (\text{A.3.1})$$

and

$$\int_S \mathbf{y} \times \llbracket \boldsymbol{\sigma}(\boldsymbol{\chi}^{iR}, \eta^{iR}) \mathbf{n} \rrbracket ds = 0. \quad (\text{A.3.2})$$

In similar fashion, we denote by $\{\boldsymbol{\chi}^i, \eta^i\}$ the solution of auxiliary local problem in free space

$$\left\{ \begin{array}{ll} -\mu(\mathbf{y}) \Delta \boldsymbol{\chi}^i + \nabla \eta^i = \mathbf{0} & \text{in } \mathbb{R}^3 \setminus S, \\ \operatorname{div} \boldsymbol{\chi}^i = 0 & \text{in } \mathbb{R}^3, \\ \llbracket \boldsymbol{\chi}^i \rrbracket = \mathbf{0} & \text{on } S, \\ \boldsymbol{\chi}^i = -\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y} & \text{on } S, \\ \boldsymbol{\chi}^i \rightarrow \mathbf{0} & \text{at } \infty, \end{array} \right. \quad (\text{A.3.3})$$

and

$$\int_S \mathbf{y} \times \llbracket \boldsymbol{\sigma}(\boldsymbol{\chi}^i, \eta^i) \mathbf{n} \rrbracket ds = 0. \quad (\text{A.3.4})$$

Existence and uniqueness of solutions to the above problems is handled in a similar fashion as in (A.2.1). In what follows we are interested in some of the properties of the solutions to the above problems.

LEMMA A.3.1. $\boldsymbol{\chi}^i$ and η^i satisfy the following pointwise estimates for $|\mathbf{x}| > R_0$

$$|D^\alpha \boldsymbol{\chi}^i(\mathbf{y})| \leq \frac{C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^{1+|\alpha|}} \text{ for } |\alpha| \leq 2,$$

$$|D^\alpha \eta^i(\mathbf{y})| \leq \frac{C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^{2+|\alpha|}} \text{ for } |\alpha| \leq 1.$$

Proof. The proof is the same as in [37], albeit with some minor adjustments with a few more steps included for completion. We first remark that for every smooth enough boundary S of the droplet we have $\|\nabla \boldsymbol{\chi}^i\|_{L^2(\mathbb{R}^3)^3} \leq C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}$. This is a consequence of the fact that for every rotation plus translation on S , $\|\nabla \boldsymbol{\chi}^i\|_{L^2(\mathbb{R}^3)^3}$ is a norm on \mathbb{R}^3 for every S . Combining this with REMARK A.2.4 we have the desired estimate.

In what follows we transform problem (A.3.3) into a problem in free space and we try to obtain estimates of the solution χ^i . Since the fundamental solution to a Stokes system behaves similarly as the fundamental solution to Laplace's equation at infinity, we expect χ^i to behave the same way.

Define,

$$\phi = \begin{cases} 0 & \text{in a neighborhood of } Y \\ 1 & \text{outside } B_{R_0} \end{cases}, \quad \phi \in C_0^\infty(\mathbb{R}^3).$$

If we multiply (A.3.3) by ϕ we obtain the following problem

$$\begin{cases} -\mu_2 \Delta (\phi \chi^i) + \nabla (\phi \eta^i) = \mathbf{f} & \text{in } \mathbb{R}^3 \\ \operatorname{div}(\phi \chi^i) = g & \text{in } \mathbb{R}^3 \end{cases} \quad (\text{A.3.5})$$

where $\mathbf{f} = -\mu_2(\Delta \phi) \chi^i - \mu_2 \nabla \chi^i \nabla \phi + \eta^i \nabla \phi$ and $g = \nabla \phi \cdot \chi^i$. We can immediately infer the following estimates for \mathbf{f} and g

$$\|\mathbf{f}\|_{L^2(\mathbb{R}^3)^3} \leq C(\|\chi^i\|_{H^1(B_{R_0})^3} + \|\eta^i\|_{L^2(B_{R_0})})$$

$$\|g\|_{L^2(\mathbb{R}^3)} \leq C \|\chi^i\|_{L^2(B_{R_0})^3}$$

Moreover, using the Gagliardo-Nirenberg-Sobolev (G-N-S) inequality we have the following estimate

$$\|\chi^i\|_{L^6(\mathbb{R}^3)^3} \leq C \|\nabla \chi^i\|_{L^2(\mathbb{R}^3)^{3 \times 3}}.$$

Combining the estimate at the beginning of the proof and the G-N-S inequality we obtain

$$\|\chi^i\|_{L^2(B_{R_0})^3} \leq C \|\nabla \chi^i\|_{L^2(\mathbb{R}^3)^3} \leq C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}.$$

Next we will obtain L^1 estimates on the data \mathbf{f} and g and then apply the Fourier transform to problem (A.3.5) to compute its solution.

From the computations above we immediately get

$$\|\mathbf{f}\|_{L^1(B_{R_0})^3} \leq C(\|\chi^i\|_{H^1(B_{R_0})^3} + \|\eta^i\|_{L^2(B_{R_0})}) \leq C(\|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3} + \|\eta^i\|_{L^2(B_{R_0})})$$

and

$$\|g\|_{L^1(B_{R_0})} \leq C \|\chi^i\|_{H^1(B_{R_0})^3} \leq C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}$$

For any fixed $R > R_0$, $\eta^i - \oint_{B_R} \eta^i d\mathbf{x}$ has mean zero over B_R . Using Lemma 5.5 in [37] there exists a $\phi_R \in H_0^1(B_R \setminus Y)^3$ such that $\operatorname{div} \phi_R = \eta^i - \oint_{B_R} \eta^i d\mathbf{y}$ and $\|\nabla \phi_R\|_{L^2(B_R \setminus Y; \mathbb{R}^3)} \leq C \left\| \eta^i - \oint_{B_R} \eta^i d\mathbf{y} \right\|_{L^2(B_R)}$. Applying the Poincare inequality we get,

$$\|\nabla \phi_R\|_{H_0^1(B_R \setminus Y)^3} \leq C \left\| \eta^i - \int_{B_R} \eta^i d\mathbf{y} \right\|_{L^2(B_R)}$$

Using ϕ_R as a test function in (A.3.3), we get

$$\int_{B_R} \nabla \chi^i : \nabla \phi_R d\mathbf{y} = \int_{B_R} \eta^i \operatorname{div} \phi_R d\mathbf{y}$$

where $\operatorname{div} \phi_R = \eta^i - \int_{B_R} \eta^i d\mathbf{y}$.

Since $C \int_{B_R} (\eta^i - \int_{B_R} \eta^i d\mathbf{y}) d\mathbf{y} = 0$ for any constant C , we write the last equation as

$$\int_{B_R} \nabla \chi^i : \nabla \phi_R d\mathbf{y} = \int_{B_R} (\eta^i - \int_{B_R} \eta^i d\mathbf{x})^2 d\mathbf{y}$$

Applying the Cauchy-Schwartz inequality we obtain

$$\left\| \eta^i - \int_{B_R} \eta^i d\mathbf{y} \right\|_{L^2(B_R)} \leq C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}$$

Moreover,

$$\left| \int_{B_R} \eta^i d\mathbf{y} \right| \leq \frac{1}{|B_R|^{1/2}} \|\eta^i\|_{L^2(\mathbb{R}^3)} \rightarrow 0$$

as $R \rightarrow \infty$. Thus,

$$\|\eta^i\|_{L^2(\mathbb{R}^3)} \leq C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}$$

Therefore, the estimate for \mathbf{f} becomes,

$$\|\mathbf{f}\|_{L^1(B_{R_0})^3} \leq C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}$$

We remark that the solution of (A.3.5) is zero in a neighborhood of Y and smooth away from the boundary, S . Denote by $\mathbf{v} = \phi \chi^i$ and $p = \phi \eta^i$, then problem (A.3.5) becomes

$$\begin{cases} -\mu_2 \Delta \mathbf{v} + \nabla p = \mathbf{f} & \text{in } \mathbb{R}^3 \\ \operatorname{div} \mathbf{v} = g & \text{in } \mathbb{R}^3 \end{cases} \quad (\text{A.3.6})$$

Applying the Fourier transform we obtain

$$\begin{aligned} -\mu_2 |\boldsymbol{\kappa}|^2 \hat{\mathbf{v}}(\boldsymbol{\kappa}) - i\boldsymbol{\kappa} \hat{p}(\boldsymbol{\kappa}) &= \hat{\mathbf{f}}(\boldsymbol{\kappa}) \\ -i\boldsymbol{\kappa} \cdot \hat{\mathbf{v}}(\boldsymbol{\kappa}) &= \hat{g}(\boldsymbol{\kappa}) \end{aligned} \quad (\text{A.3.7})$$

where $\hat{\mathbf{v}}(\boldsymbol{\kappa}) = \mathcal{F}\{\mathbf{v}(\mathbf{y}); \boldsymbol{\kappa}\}$, $\hat{p}(\boldsymbol{\kappa}) = \mathcal{F}\{p(\mathbf{y}); \boldsymbol{\kappa}\}$, $\hat{\mathbf{f}}(\boldsymbol{\kappa}) = \mathcal{F}\{\mathbf{f}(\mathbf{y}); \boldsymbol{\kappa}\}$ and $\hat{g}(\boldsymbol{\kappa}) = \mathcal{F}\{g(\mathbf{y}); \boldsymbol{\kappa}\}$

Dotting the first equation of system (A.3.7) by $-i\boldsymbol{\kappa}$ we obtain

$$\hat{p}(\boldsymbol{\kappa}) = \mu_2 \hat{g}(\boldsymbol{\kappa}) + i \frac{\boldsymbol{\kappa} \cdot \hat{\mathbf{f}}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^2}$$

and

$$\hat{\mathbf{v}}(\boldsymbol{\kappa}) = \frac{1}{\mu_2} \frac{1}{|\boldsymbol{\kappa}|^2} \hat{\mathbf{f}}(\boldsymbol{\kappa}) + i \frac{\hat{g}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^2} \boldsymbol{\kappa} - \frac{1}{\mu_2} \frac{1}{|\boldsymbol{\kappa}|^2} \frac{\boldsymbol{\kappa} \cdot \hat{\mathbf{f}}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^2} \boldsymbol{\kappa}$$

Taking the inverse Fourier transform for the pressure first we have

$$\begin{aligned} p(\mathbf{y}) &= \mathcal{F}^{-1}\{\hat{p}(\boldsymbol{\kappa}); \boldsymbol{\kappa}\} \\ &= \mathcal{F}^{-1}\{\mu_2 \hat{g}(\boldsymbol{\kappa}); \boldsymbol{\kappa}\} + \mathcal{F}^{-1}\left\{i \frac{\boldsymbol{\kappa} \cdot \hat{\mathbf{f}}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^2}; \boldsymbol{\kappa}\right\} \\ &= \mu_2 g(\mathbf{y}) + \sum_{j=1}^3 \frac{x_j}{4\pi |\mathbf{y}|^3} * f_j(\mathbf{y}) \\ &= \mu_2 g(\mathbf{y}) + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(\mathbf{y} - \mathbf{z}) \cdot \mathbf{f}(\mathbf{y})}{|\mathbf{y} - \mathbf{z}|^3} d\mathbf{z} \end{aligned}$$

Next we invert $\hat{\mathbf{v}}(\boldsymbol{\kappa})$. Inverting each term separately we obtain,

$$\begin{aligned} \mathcal{F}^{-1}\left\{\frac{1}{\mu_2 |\boldsymbol{\kappa}|^2} \hat{\mathbf{f}}(\boldsymbol{\kappa}); \mathbf{y}\right\} &= \frac{1}{\mu_2} \frac{1}{4\pi |\mathbf{y}|} * \mathbf{f}(\mathbf{y}) = \frac{1}{\mu_2} \int_{\mathbb{R}^3} \frac{1}{4\pi |\mathbf{y} - \mathbf{z}|} \mathbf{f}(\mathbf{z}) d\mathbf{z} \\ \mathcal{F}^{-1}\left\{i \frac{\hat{g}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^2} \boldsymbol{\kappa}; \mathbf{y}\right\} &= \frac{1}{4\pi} \frac{\mathbf{y}}{|\mathbf{y}|^3} * g(\mathbf{y}) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{g(\mathbf{z})}{|\mathbf{y} - \mathbf{z}|^3} (\mathbf{y} - \mathbf{z}) d\mathbf{z} \end{aligned}$$

To invert the last term we make use of the formula

$$-\frac{\mathbf{y} \otimes \mathbf{y}}{8\pi |\mathbf{y}|^3} + \frac{\mathbf{I}}{8\pi |\mathbf{y}|} = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{\boldsymbol{\kappa} \otimes \boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \exp^{-i\boldsymbol{\kappa} \cdot \mathbf{y}} d\boldsymbol{\kappa}$$

and we obtain

$$\mathcal{F}^{-1}\left\{-\frac{1}{\mu_2} \frac{\boldsymbol{\kappa} \cdot \hat{\mathbf{f}}(\boldsymbol{\kappa})}{|\boldsymbol{\kappa}|^4} \boldsymbol{\kappa}; \mathbf{y}\right\} = \frac{1}{\mu_2} \int_{\mathbb{R}^3} \frac{(\mathbf{y} - \mathbf{z}) \cdot \mathbf{f}(\mathbf{z})}{8\pi |\mathbf{y} - \mathbf{z}|^3} (\mathbf{y} - \mathbf{z}) d\mathbf{z} - \frac{1}{\mu_2} \int_{\mathbb{R}^3} \frac{\mathbf{f}(\mathbf{z})}{8\pi |\mathbf{y} - \mathbf{z}|} d\mathbf{z}$$

Therefore,

$$\begin{aligned} \mathbf{v}(\mathbf{y}) &= \frac{1}{\mu_2} \int_{\mathbb{R}^3} \frac{1}{4\pi |\mathbf{y} - \mathbf{z}|} \mathbf{f}(\mathbf{z}) d\mathbf{z} + \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{g(\mathbf{z})}{|\mathbf{y} - \mathbf{z}|^3} (\mathbf{y} - \mathbf{z}) d\mathbf{z} \\ &\quad + \frac{1}{\mu_2} \int_{\mathbb{R}^3} \frac{(\mathbf{y} - \mathbf{z}) \cdot \mathbf{f}(\mathbf{z})}{8\pi |\mathbf{y} - \mathbf{z}|^3} (\mathbf{y} - \mathbf{z}) d\mathbf{z}. \end{aligned}$$

Since $|\mathbf{y}| > R_0$ we have $\phi(\mathbf{y}) = 1$ and furthermore, given the compact support of \mathbf{f} and g in B_{R_0} we get the following estimates:

$$|\chi^i| \leq c_1 \int_{B_{R_0}} \frac{|\mathbf{f}(\mathbf{z})|}{|\mathbf{y} - \mathbf{z}|} d\mathbf{z} + c_2 \int_{B_{R_0}} \frac{|g(\mathbf{z})|}{|\mathbf{y} - \mathbf{z}|^3} |\mathbf{y} - \mathbf{z}| d\mathbf{z} + c_3 \int_{B_{R_0}} \frac{|(\mathbf{y} - \mathbf{z}) \cdot \mathbf{f}(\mathbf{z})|}{8\pi |\mathbf{y} - \mathbf{z}|^3} |\mathbf{y} - \mathbf{z}| d\mathbf{z}$$

Applying Holder's inequality we obtain,

$$|\chi^i| \leq c_4 \|\mathbf{f}\|_{L^1(B_{R_0})^3} + c_5 \|g\|_{L^1(B_{R_0})} \leq C \|-\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}.$$

Next we compute the derivatives of χ^i .

$$\begin{aligned} \frac{\partial \chi_m^i(\mathbf{y})}{\partial y_k} &= \frac{1}{8\pi\mu_2} \int_{B_{R_0}} \frac{-f_m(\mathbf{y})(y_m - z_m)}{|\mathbf{y} - \mathbf{z}|^3} d\mathbf{z} + \frac{1}{4\pi} \int_{B_{R_0}} \frac{g(\mathbf{z})\delta_{km}}{|\mathbf{y} - \mathbf{z}|^3} d\mathbf{z} \\ &+ \frac{1}{8\pi\mu_2} \int_{B_{R_0}} \frac{\mathbf{f}(\mathbf{z}) \cdot (\mathbf{y} - \mathbf{z})}{|\mathbf{y} - \mathbf{z}|^3} \delta_{km} d\mathbf{z} + \frac{1}{8\pi\mu_2} \int_{B_{R_0}} \frac{f_k(\mathbf{z})}{|\mathbf{y} - \mathbf{z}|^3} d\mathbf{z} \\ &- \frac{1}{8\pi\mu_2} \int_{\mathbb{R}^3} \frac{3(y_k - z_k)(\mathbf{y} - \mathbf{z}) \cdot \mathbf{f}(\mathbf{z})}{|\mathbf{y} - \mathbf{z}|^5} d\mathbf{z}. \end{aligned}$$

Taking absolute values, using properties of the integral, and carrying out some further computations one obtains:

$$\left| \frac{\partial \chi_m^i(\mathbf{y})}{\partial x_k} \right| = \frac{c \|-\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^2}.$$

In exactly the same manner we obtain estimates of the derivatives of χ^i and η^i which look as follows:

$$|D^\alpha \chi^i(\mathbf{y})| = \frac{c \|-\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^{1+|\alpha|}} \text{ for } |\alpha| \leq 2.$$

$$|D^\alpha \eta^i(\mathbf{y})| = \frac{c \|-\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^{2+|\alpha|}} \text{ for } |\alpha| \leq 1.$$

□

LEMMA A.3.2. *There exists a constant C such that for every $R > R_0$ we have*

$$\|\sigma(\chi^{iR}, \eta^{iR})\|_{L^2(\partial B_R)^{3 \times 3}} \leq \frac{C \|-\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{R^2}.$$

Proof. Take ϕ to be a smooth cut-off function defined to be zero in neighborhood of Y and 1 outside B_{R_0} . The pair $\{\phi\chi^{iR}, \phi\eta^{iR}\}$ satisfies the following Stokes system

$$\begin{cases} -\mu_2 \Delta (\phi\chi^{iR}) + \nabla(\phi\eta^{iR}) = \mathbf{f}^R & \text{in } B_R, \\ \operatorname{div}(\phi\chi^{iR}) = g^R & \text{in } B_R, \\ \phi\chi^{iR} = \mathbf{0} & \text{on } \partial B_R. \end{cases}$$

If $\{\tilde{\chi}^i, \tilde{\eta}^i\}$ is the solution pair for the corresponding Stokes system in free space then by the previous lemma we have the following pointwise estimates, for $|\mathbf{y}| > R_0$

$$|D^\alpha \tilde{\chi}^i(x)| \leq \frac{C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^{1+|\alpha|}} \text{ for } |\alpha| \leq 2,$$

$$|D^\alpha \tilde{\eta}^i(x)| \leq \frac{C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{|\mathbf{y}|^{2+|\alpha|}} \text{ for } |\alpha| \leq 1.$$

Using the above estimates for any $R > R_0$ we get

$$\|D^\alpha \tilde{\chi}^i\|_{L^2(B_{2R} \setminus B_R)^3} \leq \frac{C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{R^{2|\alpha|-1}},$$

$$\|D^\alpha \tilde{\eta}^i\|_{L^2(B_{2R} \setminus B_R)} \leq \frac{C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{R^{2|\alpha|+1}},$$

and furthermore we obtain the following bound for the stress

$$\begin{aligned} \|\boldsymbol{\sigma}(\tilde{\chi}^i, \tilde{\eta}^i)\|_{L^2(\partial B_R)^{3 \times 3}} &= \|2\mu_2 e(\tilde{\chi}^i) - \tilde{\eta}^i I\|_{L^2(\partial B_R)^{3 \times 3}} \\ &\leq C\{\|D\tilde{\chi}^i\|_{L^2(\partial B_R)^{3 \times 3}} + \|\tilde{\eta}^i\|_{L^2(\partial B_R)}\} \\ &\leq \frac{C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}}{R^2}. \end{aligned}$$

After changing variables we obtain

$$\left\| \tilde{\chi}^i \left(\frac{R}{R_0} \cdot \right) \right\|_{H^2(B_{2R_0} \setminus B_{R_0})^3} \leq C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3} \frac{1}{R},$$

and

$$\left\| \tilde{\eta}^i \left(\frac{R}{R_0} \cdot \right) \right\|_{H^1(B_{2R_0} \setminus B_{R_0})} \leq C \|\mathbf{-e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3} \frac{1}{R^2}.$$

Define, $\mathbf{u}^i = \phi\chi^{iR} \left(\frac{R}{R_0} \mathbf{y} \right) - \phi\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{y} \right)$, and $p^i = \frac{R}{R_0} \phi\eta^{iR} \left(\frac{R}{R_0} \mathbf{y} \right) - \frac{R}{R_0} \phi\tilde{\eta}^i \left(\frac{R}{R_0} \mathbf{y} \right)$. The pair $\{\mathbf{u}^i, p^i\}$ satisfies the following Stokes system

$$\begin{cases} -\mu_2 \Delta \mathbf{u}^i + \nabla p^i = \mathbf{0} & \text{in } B_{R_0}, \\ \operatorname{div} \mathbf{u}^i = 0 & \text{in } B_{R_0}, \\ \mathbf{u}^i = -\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{y} \right) & \text{on } \partial B_{R_0}. \end{cases}$$

The regularity results in [58] (Prop. 2.2) yield

$$\begin{aligned} \|\mathbf{u}^i\|_{H^2(B_{R_0})^3} + \|p^i\|_{H^1(B_{R_0})} &\leq \left\| -\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{y} \right) \right\|_{H^{3/2}(\partial B_{R_0})^3} \\ &\leq C \left\| -\tilde{\chi}^i \left(\frac{R}{R_0} \mathbf{y} \right) \right\|_{H^2(B_{2R_0} \setminus B_{R_0})^3} \\ &\leq \frac{1}{R} C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}. \end{aligned}$$

Using the results above we can compute

$$\begin{aligned} \|\boldsymbol{\sigma}(\mathbf{u}^i, p^i)\|_{L^2(\partial B_{R_0})^{3 \times 3}} &= \|2\mu_2 e(\mathbf{u}^i) - p^i I\|_{L^2(\partial B_{R_0})^{3 \times 3}} \\ &\leq C \left\{ \|\mathbf{u}^i\|_{H^2(B_{R_0})^3} + \|p^i\|_{H^1(B_{R_0})} \right\} \\ &\leq \frac{1}{R} C \|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3}. \end{aligned}$$

Applying a change of variable yet again we obtain

$$\|\boldsymbol{\sigma}(\mathbf{u}^i, p^i)\|_{L^2(\partial B_{R_0})^{3 \times 3}} = \|\boldsymbol{\sigma}(\boldsymbol{\chi}^{iR}, \eta^{iR}) - \boldsymbol{\sigma}(\tilde{\chi}^i, \tilde{\eta}^i)\|_{L^2(\partial B_{R_0})^{3 \times 3}}.$$

Hence,

$$\|\boldsymbol{\sigma}(\boldsymbol{\chi}^{iR}, \eta^{iR})\|_{L^2(\partial B_R)^{3 \times 3}}^2 \leq C \left(\|\mathbf{e}_i + \boldsymbol{\omega} \times \mathbf{y}\|_{L^2(S)^3} \frac{1}{R} \right)^2.$$

□

A.4 Derivation of the stress jump on a droplet surface

In this sections of the appendix we provide a justification for the stress jump formula used in Chapter 2 (see [25], [41]). If we balance the linear momentum of an elemental surface area S in a viscous time stationary flow (see figure A.1) we obtain

$$0 = \int_S \llbracket \boldsymbol{\sigma} \mathbf{n} \rrbracket ds + \oint_C \lambda \mathbf{t} d\ell \quad (\text{A.4.1})$$

where λ is some function that depends on the position on S .

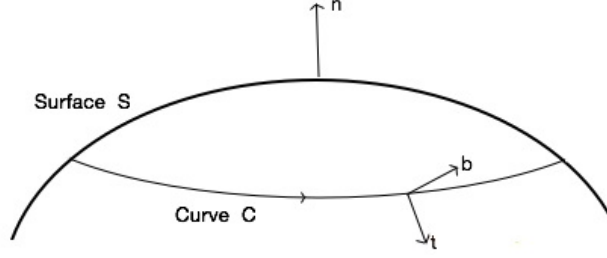


Figure A.1: Surface tension on an elemental surface

Elementary results from vector calculus tell us that $\mathbf{n} = \mathbf{t} \times \mathbf{b}$ and $\mathbf{t} = \mathbf{b} \times \mathbf{n}$. We can then rewrite the last term of (A.4.1) as

$$\oint_C \lambda \mathbf{t} d\ell = - \oint_C \lambda \mathbf{n} \times \mathbf{b} d\ell$$

Since S is a smooth orientable surface bounded by a simple closed curve C then by Stokes theorem we have,

$$\oint_C \mathbf{F} \cdot \mathbf{b} d\ell = \int_S \text{curl } \mathbf{F} \cdot \mathbf{n} ds$$

for any C^1 vector field \mathbf{F} , where \mathbf{n} is the unit normal to S and \mathbf{b} is the unit tangent vector to C that points in the positive direction. For any vector field \mathbf{F} of the form $\mathbf{F} = \mathbf{f} \times \mathbf{m}$, where \mathbf{m} is any non-zero, constant vector field, the above formula reduces to

$$-\mathbf{m} \cdot \oint_C \mathbf{f} \times \mathbf{b} d\ell = \int_S \text{curl } \mathbf{f} \times \mathbf{m} \cdot \mathbf{n} ds \quad (\text{A.4.2})$$

Consider, the integrand on the right hand side,

$$\begin{aligned} \text{curl } \mathbf{f} \times \mathbf{m} &= \epsilon_{ijk} \epsilon_{kpq} \frac{\partial}{\partial x_j} (f_p m_q) \\ &= (\delta_{ip} \delta_{jp} - \delta_{iq} \delta_{jp}) \frac{\partial f_p}{\partial x_j} m_q \\ &= \frac{\partial f_i}{\partial x_j} m_j - \frac{\partial f_j}{\partial x_i} m_i \end{aligned}$$

Thus, $\text{curl } \mathbf{f} \times \mathbf{m} \cdot \mathbf{n} = -\mathbf{m} \cdot (\mathbf{n} \text{div } \mathbf{f} - \mathbf{n} \cdot \nabla \mathbf{f})$ and (A.4.2) can be re-written as

$$\oint_C \mathbf{f} \times \mathbf{b} d\ell = \int_S (\mathbf{n} \text{div } \mathbf{f} - \mathbf{n} \cdot \nabla \mathbf{f}) \cdot \mathbf{m} ds.$$

If we let $\mathbf{f} = \lambda \mathbf{n}$ then $\lambda \mathbf{n} \times \mathbf{b} = -\lambda \mathbf{t}$, $\text{div}(\lambda \mathbf{n}) = (\nabla \lambda \cdot \mathbf{n}) \mathbf{n} + \lambda (\text{div} \mathbf{n}) \mathbf{n}$, and $\nabla(\lambda \mathbf{n}) \cdot \mathbf{n} = \nabla \lambda + \lambda (\nabla \mathbf{n} \cdot \mathbf{n})$. We note that $(\nabla \mathbf{n} \cdot \mathbf{n}) = 0$ since $(\nabla \mathbf{n} \cdot \mathbf{n}) = \frac{\partial n_i}{\partial x_j} n_i = (1/2) \frac{\partial}{\partial x_j} (n_i n_i) = 0$. Hence,

$$\begin{aligned}
-\oint_C \lambda \mathbf{t} d\ell &= -\int_S (\nabla \lambda - \mathbf{n}(\mathbf{n} \cdot \nabla) \lambda - \lambda (\operatorname{div} \mathbf{n}) \mathbf{n}) ds \\
&= -\int_S (\nabla_S \lambda - \lambda (\operatorname{div} \mathbf{n}) \mathbf{n}) ds
\end{aligned}$$

From (A.4.2) we obtain,

$$\int_S \llbracket \sigma \mathbf{n} \rrbracket ds = \int_S (-\nabla_S \lambda + \lambda (\operatorname{div} \mathbf{n}) \mathbf{n}) ds$$

for every elemental surface S . Thus,

$$\llbracket \sigma \mathbf{n} \rrbracket = \nabla_S \lambda - \lambda \kappa \mathbf{n} \quad \text{on } S,$$

where $\kappa = \operatorname{div} \mathbf{n}$ is the mean curvature.

A.5 Definitions and properties of Γ -convergence

DEFINITION A.5.1. Let (X, τ) be a first countable topological space and $(F^\epsilon)_\epsilon$ a sequence of functions from X into $\bar{\mathbb{R}}$. We say F^ϵ is Γ -convergent to F as $\epsilon \rightarrow 0$ and write $F = \Gamma - \lim_{\epsilon \rightarrow 0} F^\epsilon$, if

$$\begin{aligned}
\forall x_\epsilon \xrightarrow{\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) &\leq \liminf_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon) \\
\exists x_\epsilon \xrightarrow{\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) &\geq \limsup_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon)
\end{aligned}$$

THEOREM A.5.2. Let (X, τ) be a first countable topological space and $(F^\epsilon)_\epsilon$ a sequence of functions from X into $\bar{\mathbb{R}}$. The following statements are equivalent:

(i)

$$\begin{aligned}
(\Gamma 1) \quad \forall x_\epsilon \xrightarrow{\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) &\leq \liminf_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon) \\
(\Gamma 2) \quad \exists x_\epsilon \xrightarrow{\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) &\geq \limsup_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon)
\end{aligned}$$

(ii)

$$\begin{aligned}
\forall x_\epsilon \xrightarrow{\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) &\leq \liminf_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon) \\
\exists x_\epsilon \xrightarrow{\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) &= \lim_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon)
\end{aligned}$$

THEOREM A.5.3. Let (X, τ) be a first countable topological space and assume $F = \Gamma - \lim_{\epsilon \rightarrow 0} F^\epsilon$. Assume further that $(x_\epsilon)_\epsilon$ is a sequence in X such that

$$F^\epsilon(x_\epsilon) \leq \inf F^\epsilon(x) + \epsilon$$

then for any convergent subsequence $(x_{\epsilon_n}) \subset (x_\epsilon)$, $x_{\epsilon_n} \xrightarrow{\tau} x$ we have:

$$F(x) = \min_{y \in X} F(y) \text{ and } F^{\epsilon_n}(x_{\epsilon_n}) \rightarrow F(x).$$

THEOREM A.5.4. Let (X, τ) be a first countable topological space and $(F^\epsilon)_\epsilon$ a sequence of functions from X into \mathbb{R} such that F^ϵ is Γ -convergent with respect to the relevant topology. Then, for every continuous function $G : X \rightarrow \mathbb{R}$, the sequence $(F^\epsilon + G)_\epsilon$ is still Γ -convergent and

$$\Gamma - \lim_{\epsilon \rightarrow 0} (F^\epsilon + G) = \Gamma - \lim_{\epsilon \rightarrow 0} F^\epsilon + G.$$

DEFINITION A.5.5. Let (X, τ) be a first countable topological space and $(F^\epsilon)_\epsilon$ a sequence of functions from X into \mathbb{R} . The sequence (F^ϵ) is said to Mosco converge to another functional $F : X \mapsto \mathbb{R}$ and write $F = M - \lim_{\epsilon \rightarrow 0} F^\epsilon$ if the following two conditions hold:

$$(M1) \forall x_\epsilon \xrightarrow{w-\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) \leq \liminf_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon)$$

$$(M2) \exists x_\epsilon \xrightarrow{s-\tau} x, \text{ as } \epsilon \rightarrow 0 \quad F(x) \geq \limsup_{\epsilon \rightarrow 0} F^\epsilon(x_\epsilon)$$

In what follows we make precise the statement that relates M and Γ convergence in the time dependent emulsion flow problem in Chapter 1. Consider the functionals E^ϵ and E defined in (1.6.1) and (1.6.2) respectively, then

THEOREM A.5.6. $E = M - \lim_{\epsilon \rightarrow 0} E^\epsilon$ in $L^2(\Omega)^3$ if and only if $E = \Gamma - \lim_{\epsilon \rightarrow 0} E^\epsilon$ in $w - H_0^1(\Omega)^3$.

Proof. Assume that $E = M - \lim_{\epsilon \rightarrow 0} E^\epsilon$ in $L^2(\Omega)^3$. To prove $(\Gamma 1)$ consider $\mathbf{u}^\epsilon, \mathbf{u} \in H_0^1(\Omega)^3$ with $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $H_0^1(\Omega)^3$. Then, $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $L^2(\Omega)^3$ which implies $\liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u})$.

To prove $(\Gamma 2)$, assume first that $E(\mathbf{u}) < +\infty$ then choose the \mathbf{u}^ϵ from $(M2)$. Thus, $\mathbf{u}^\epsilon \rightarrow \mathbf{u}$ in $L^2(\Omega)^3$ and $\limsup_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \leq E(\mathbf{u})$. This implies that $E(\mathbf{u}^\epsilon)$ is bounded and by Korn's inequality in $H_0^1(\Omega)^3$ we obtain that $\|\mathbf{u}^\epsilon\|_{H_0^1(\Omega)^3}$ is bounded. Hence, $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $H_0^1(\Omega)^3$. If $E(\mathbf{u}) = +\infty$ choose $\mathbf{u}^\epsilon = \mathbf{u}$, then $\mathbf{u}^\epsilon \rightarrow \mathbf{u}$ in $H_0^1(\Omega)^3$ and $\limsup_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u}) = +\infty$.

Assume now that $E = \Gamma - \lim_{\epsilon \rightarrow 0} E^\epsilon$ in $w - H_0^1(\Omega)^3$. To prove $(M1)$ consider $\mathbf{u}^\epsilon, \mathbf{u} \in L^2(\Omega)^3$ with $\mathbf{u}^\epsilon \rightharpoonup \mathbf{u}$ in $L^2(\Omega)^3$. If $\liminf_{\epsilon \rightarrow 0} \|\mathbf{u}^\epsilon\|_{H_0^1(\Omega)^3} = +\infty$ then $E^\epsilon(\mathbf{u}^\epsilon) \rightarrow +\infty$ and $(M1)$ follows in this case. If for some $\|\mathbf{u}^{\epsilon_k}\|_{H_0^1(\Omega)^3}$ is bounded then $\mathbf{u}^{\epsilon_k} \rightharpoonup \mathbf{u}$ in $H_0^1(\Omega)^3$ and by $(\Gamma 1)$ we have $\liminf_{\epsilon \rightarrow 0} E^\epsilon(\mathbf{u}^\epsilon) \geq E(\mathbf{u})$.

To prove $(M2)$, if $\mathbf{u} \in L^2(\Omega)^3 \setminus H_0^1(\Omega)^3$ pick $\mathbf{u}^\epsilon = \mathbf{u}$ and the result follows in a trivial way. If $\mathbf{u} \in H_0^1(\Omega)^3$ then choose \mathbf{u}^ϵ from $(\Gamma 2)$. By the Rellich compact embedding theorem of $H_0^1(\Omega)^3 \subset\subset L^2(\Omega)^3$, up to a subsequence still denote by \mathbf{u}^ϵ , $(M2)$ follows.

A.6 Finite element implementation using the viscosity and penalization methods

In this section we describe how to implement the FEM using the viscosity and penalization methods to carry out the simulations for Chapter 3 using **FreeFem++**.

A.6.1 The viscosity method to enforce zero average

Consider the following periodic transmission elliptic problem on the cell Y ,

$$\begin{aligned} -\frac{\partial}{\partial y_i} \left(\mu \left(-\frac{\partial u^k}{\partial y_i} + \delta_{ik} \right) \right) &= 0 \text{ in } Y, \\ -\llbracket \mu \left(-\frac{\partial u^k}{\partial y_i} + \delta_{ik} \right) n_i \rrbracket &= 0 \text{ on } S, \\ u^k \text{ is } Y\text{-periodic, } \mathcal{M}_Y(u^k) &= 0. \end{aligned} \tag{A.6.1}$$

Here, $\mathcal{M}(u^k) = \frac{1}{|Y|} \int_Y u^k d\mathbf{y}$, $\mu = \mu_1$ inside the inclusion and $\mu = \mu_2$ outside the inclusion and S indicates the interface (see Figure 1).

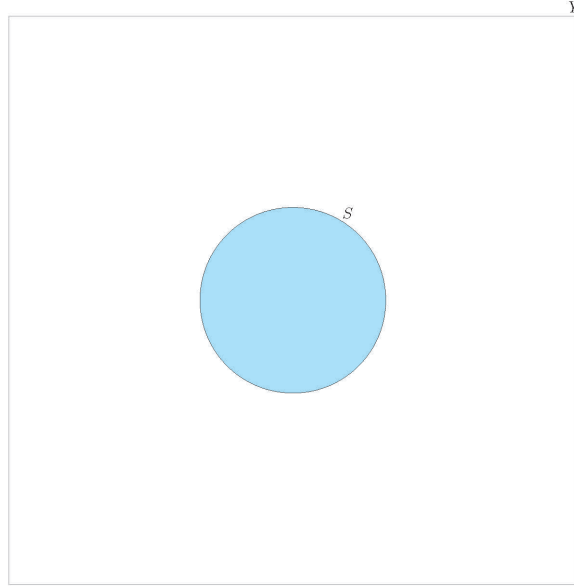


Figure A.2: The cell problem

Define $W_{per}(Y) = \{w \in H_{per}^1(Y) \mid \mathcal{M}_Y(u^k) = 0\}$, then the variational formulation of (A.6.1) is

$$\begin{aligned} &\text{Find } u^k \in W_{per}(Y) \text{ such that} \\ &\int_Y \mu \nabla u^k \cdot \nabla v \, d\mathbf{y} - \int_Y \mu e_k \cdot \nabla v \, d\mathbf{y} = 0 \text{ for all } v \in W_{per}(Y). \end{aligned} \quad (\text{A.6.2})$$

Existence and uniqueness follows from a combination of the Lax-Milgram lemma, Poincare and Holder inequalities. Equivalently, we can write (A.6.2) as

$$\begin{aligned} &\text{Find } u^k \in H_{per}^1(Y) \text{ such that} \\ &J(u^k) = \min_{w \in H_{per}^1(Y)} J(w). \end{aligned} \quad (\text{A.6.3})$$

$$\text{where } J(w) = \frac{1}{2} \int_Y \mu |\nabla w|^2 \, d\mathbf{y} - \int_Y \mu e_k \cdot \nabla w \, d\mathbf{y} + I_{W_{per}(Y)}(w).$$

The space $W_{per}(Y)$ is not an easy space to implement the FEM. Thus, we consider the approximating problem

$$\begin{aligned} &\text{Find } u^{\epsilon k} \in H_{per}^1(Y) \text{ such that} \\ &J^\epsilon(u^{\epsilon k}) = \min_{w \in H_{per}^1(Y)} J^\epsilon(w). \end{aligned} \quad (\text{A.6.4})$$

$$\text{where } J^\epsilon(w) = \frac{1}{2} \int_Y \mu |\nabla w|^2 \, d\mathbf{y} + \frac{\epsilon}{2} \int_Y |w|^2 \, d\mathbf{y} - \int_Y \mu e_k \cdot \nabla w \, d\mathbf{y}.$$

The functional J^ϵ is monotone decreasing as ϵ tends to zero and therefore it Γ -converges to the functional J in $w\text{-}H_{per}^1(Y)$. By corollary 1.6.4 we have that $u^{\epsilon k} \rightarrow u^k$ in $w\text{-}H_{per}^1(Y)$. Computing the Gateaux derivative of J^ϵ we obtain,

$$\begin{aligned} &\text{Find } u^{\epsilon k} \in H_{per}^1(Y) \text{ such that} \\ &\int_Y \mu \nabla u^{\epsilon k} \cdot \nabla v \, d\mathbf{y} + \epsilon \int_Y u^{\epsilon k} v \, d\mathbf{y} - \int_Y \mu e_k \cdot \nabla v \, d\mathbf{y} = 0 \text{ for all } v \in H_{per}^1(Y). \end{aligned} \quad (\text{A.6.5})$$

Setting $v = \text{const.}$ in (A.6.5) we obtain $\mathcal{M}_Y(u^{\epsilon k}) = 0$. Since $\mathcal{M}_Y(u^k) = \lim_{\epsilon \rightarrow 0} \mathcal{M}_Y(u^{\epsilon k}) = 0$ the sequence $u^{\epsilon k}$ has all the desired properties and it is easy compute numerically using the FEM.

A.6.2 The penalization method to enforce rigid body motion

Consider the suspension of a single rigid spherical particle in a time stationary viscous fluid,

$$\begin{aligned}
-\operatorname{div} \sigma &= \mathbf{f} && \text{in } Y_f \\
\sigma_{ij} &= -p \delta_{ij} + 2 \mu e_{ij}(\mathbf{u}) \\
\operatorname{div} \mathbf{u} &= 0 && \text{in } Y_f \\
\llbracket \mathbf{u} \rrbracket &= \mathbf{0} && \text{on } S \\
e(\mathbf{u}) &= \mathbf{0} && \text{in } T \\
\mathbf{u} &= \mathbf{0} && \text{on } \partial Y
\end{aligned} \tag{A.6.6}$$

together with the balance of forces and torques,

$$\int_S \sigma \mathbf{n} ds + \int_T \mathbf{f} d\mathbf{x} = \mathbf{0}, \quad \int_S \mathbf{x} \times \sigma \mathbf{n} ds + \int_T \mathbf{x} \times \mathbf{f} d\mathbf{x} = \mathbf{0} \tag{A.6.7}$$

The variational formulation of (A.6.6)-(A.6.7) is: given $\mathbf{f} \in L^2(Y)$,

$$\begin{aligned}
&\text{find } \mathbf{u} \in V \text{ such that} \\
&a(\mathbf{u}, \mathbf{v}) = \int_Y \mathbf{f} \cdot \mathbf{v} d\mathbf{x} \text{ for all } \mathbf{v} \in V
\end{aligned} \tag{A.6.8}$$

where V is the closed subspace of $H_0^1(Y)^3$ defined by

$$V = \{\mathbf{w} \in H_0^1(Y)^3 \mid \operatorname{div} \mathbf{w} = \mathbf{0} \text{ in } Y_f, e(\mathbf{w}) = 0 \text{ in } T\}$$

and $a(\mathbf{u}, \mathbf{v}) = \int_{Y_f} 2 \mu e(\mathbf{u}) : e(\mathbf{v}) d\mathbf{x}$. The corresponding minimization problem is:

$$\begin{aligned}
&\text{find } \mathbf{u} \in H_0^1(Y)^3 \text{ such that} \\
&J(\mathbf{u}) = \min_{\mathbf{w} \in H_0^1(Y)^3} J(\mathbf{w})
\end{aligned} \tag{A.6.9}$$

where $J(\mathbf{w}) = \frac{1}{2} a(\mathbf{w}, \mathbf{w}) - (\mathbf{f}, \mathbf{v}) + I_V(\mathbf{w})$.

We can approximate the solution to problem (A.6.9) by the solving the following problem:

$$\begin{aligned}
&\text{find } \mathbf{u}^\epsilon \in H_0^1(Y)^3 \text{ such that} \\
&J^\epsilon(\mathbf{u}^\epsilon) = \min_{\mathbf{w} \in H_0^1(Y)^3} J^\epsilon(\mathbf{w})
\end{aligned} \tag{A.6.10}$$

where $J^\epsilon(\mathbf{w}) = \frac{1}{2} a(\mathbf{w}, \mathbf{w}) + \frac{1}{2\epsilon} \int_T |e(\mathbf{w})|^2 d\mathbf{x} - (\mathbf{f}, \mathbf{v}) + I_W(\mathbf{w})$ and $W = \{\mathbf{w} \in H_0^1(Y)^3 \mid \operatorname{div} \mathbf{w} = \mathbf{0} \text{ in } Y_f\}$. The functional J^ϵ is monotone increasing as ϵ tends to zero and thus it Γ -converges to J in $w\text{-}H_0^1(Y)^3$. As before we immediately obtain that $\mathbf{u}^\epsilon \rightarrow \mathbf{u}$ in $w\text{-}H_0^1(Y)^3$. Computing the Gateaux derivative of J^ϵ we obtain

$$\begin{aligned}
&\text{find } \mathbf{u}^\epsilon \in W \text{ such that} \\
&a(\mathbf{u}^\epsilon, \mathbf{v}) + \frac{1}{\epsilon} \int_T e(\mathbf{u}^\epsilon) : e(\mathbf{v}) d\mathbf{x} = \int_Y \mathbf{f} \cdot \mathbf{v} d\mathbf{x} \text{ for all } \mathbf{v} \in W
\end{aligned} \tag{A.6.11}$$

A.6.3 FEM implementation of Stokes flow in FreeFem++

Consider the following Stokes flow in a domain Ω with smooth enough boundary,

$$\begin{aligned} -\mu \Delta \mathbf{u} + \nabla p &= \mathbf{0} && \text{in } \Omega \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= \mathbf{u}_0 && \text{on } \partial\Omega \\ \int_{\Omega} p \, d\mathbf{x} &= 0 \end{aligned} \tag{A.6.12}$$

with compatibility condition $\int_{\partial\Omega} \mathbf{u}_0 \cdot \mathbf{n} \, ds = 0$

The variational formulation of (A.6.12) is:

$$\begin{aligned} \text{find } (\mathbf{u}, p) &\in H^1(\Omega)^3 \times L_0^2(\Omega) \text{ such that} \\ \int_{\Omega} \mu \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} - \int_{\Omega} p \operatorname{div} \mathbf{v} \, d\mathbf{x} &= 0 \text{ for all } \mathbf{v} \in H_0^1(\Omega)^3 \\ \int_{\Omega} q \operatorname{div} \mathbf{u} \, d\mathbf{x} &= 0 \text{ for all } q \in L_0^2(\Omega) \\ \int_{\Omega} p \, d\mathbf{x} &= 0 \end{aligned} \tag{A.6.13}$$

using the results in the appendix we can recover (A.6.12) from (A.6.13). The space $L_0^2(\Omega)$ contains all elements that have average zero. To remove this condition from the space we modify (A.6.13) the following way:

$$\begin{aligned} \text{find } (\mathbf{u}, p, c) &\in H^1(\Omega)^3 \times L^2(\Omega) \times \mathbb{R} \text{ such that} \\ \int_{\Omega} \mu \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x} - \int_{\Omega} p \operatorname{div} \mathbf{v} \, d\mathbf{x} &= 0 \text{ for all } \mathbf{v} \in H_0^1(\Omega)^3 \\ \int_{\Omega} q \operatorname{div} \mathbf{u} \, d\mathbf{x} + c \int_{\Omega} q \, d\mathbf{x} &= 0 \text{ for all } q \in L^2(\Omega) \\ \int_{\Omega} p \, d\mathbf{x} &= 0 \end{aligned} \tag{A.6.14}$$

Problem (A.6.13) is equivalent to (A.6.14). To see this assume that $(\mathbf{u}, p) \in H^1(\Omega)^3 \times L^2(\Omega)$ and satisfies (A.6.13) then it trivially satisfies (A.6.14) for $c = 0$. On the other hand if $(\mathbf{u}, p, c) \in H^1(\Omega)^3 \times L^2(\Omega) \times \mathbb{R}$ and satisfies (A.6.14) then for all $q \in L^2(\Omega)$ consider

$$\begin{aligned}
\int_{\Omega} (q - \mathcal{M}_{\Omega}(q)) \operatorname{div} \mathbf{u} \, d\mathbf{x} + c \int_{\Omega} (q - \mathcal{M}_{\Omega}(q)) \, d\mathbf{x} &= 0 \\
\int_{\Omega} (q - \mathcal{M}_{\Omega}(q)) \operatorname{div} \mathbf{u} \, d\mathbf{x} &= 0 \\
\int_{\Omega} q \operatorname{div} \mathbf{u} \, d\mathbf{x} = \tilde{q} \int_{\Omega} \operatorname{div} \mathbf{u} \, d\mathbf{x} &= 0
\end{aligned}$$

by the compatibility condition and thus we recover (A.6.13).

A.7 FreeFem++ code for Chapter 3

In this section of the appendix we give the code needed to generate the plots in Chapter 3.

```

int n=20;
border C11(t=0,1){ x=t; y=0; label=1;}
border C12(t=0,1){ x=1; y=t; label=2;}
border C13(t=0,1){ x=1-t; y=1; label=3;}
border C14(t=0,1){ x=0; y=1-t; label=4;}

// volume fraction 14%
border C(t=0,2*pi){x=0.5+0.2111*cos(t); y=0.5+0.2111*sin(t);}

mesh Th=buildmesh( C(n)+C11(n)+C12(n)+C13(n)+C14(n) );

fespace Wh(Th,P2,periodic=[[2,y],[4,y],[1,x],[3,x]]); Wh phi1, phi2, v;

Wh pl=region;
int inside=pl(0.5,0.5);
Wh xi = (region==inside);
Wh kappa = 6.7*xi + 1*(1-xi);
Wh kappa2 = 6.7*xi + 0*(1-xi);

//int rigid=Th(0.5,0.5).region, fluid=Th(0.9,0.9).region;

//////////////////////
// scaled magnetic permeability of magnetite=6.7, water=1
//////////////////////
//Wh kappa=6.7*(region==rigid)+1*(region==fluid);

solve a1(phi1,v)= int2d(Th)(kappa*(dx(phi1)*dx(v)+dy(phi1)*dy(v)))
- int2d(Th)(kappa*dx(v))
+ int2d(Th)(phi1*v*(1.e-8)); //

solve a2(phi2,v)= int2d(Th)(kappa*(dx(phi2)*dx(v)+dy(phi2)*dy(v)))
- int2d(Th)(kappa*dy(v))
+ int2d(Th)(phi2*v*(1.e-8)); //

real ms11 = int2d(Th) (kappa2*(-dx(phi1) + 1));
real ms12 = int2d(Th) (-kappa2*dx(phi2));
real ms21 = int2d(Th) (-kappa2*dy(phi1));
real ms22 = int2d(Th) (kappa2*(-dy(phi2)+1));

cout << " ms_11 is " << 0.00000123662*ms11 ;
cout << " ms_12 is " << 0.00000123662*ms12 ;
cout << " ms_21 is " << 0.00000123662*ms21 ;
cout << " ms_22 is " << 0.00000123662*ms22 ;

real m11 = int2d(Th) (kappa*(-dx(phi1) + 1));
real m12 = int2d(Th) (-kappa*dx(phi2));
real m21 = int2d(Th) (-kappa*dy(phi1));
real m22 = int2d(Th) (kappa*(-dy(phi2)+1));

cout << " m_11 is " << 0.00000123662*m11 ;
cout << " m_12 is " << 0.00000123662*m12 ;
cout << " m_21 is " << 0.00000123662*m21 ;
cout << " m_22 is " << 0.00000123662*m22 ;

fespace Vh(Th,[P2,P2],periodic=[[2,y],[4,y],[1,x],[3,x]]);
fespace Ph(Th,P1,periodic=[[2,y],[4,y],[1,x],[3,x]]);

```

```

// characteristic function one in the solid, zero otherwise.
Ph chi = 1*xi + 0*(1-xi);

macro w11 [w11,w112] //
macro w12 [w12,w122] //
macro w22 [w22,w222] //
macro v [v1,v2] //

macro e(w) [dx(w[0]),(dy(w[0])+dx(w[1]))/sqrt(2.),dy(w[1])] //
macro div(w) (dx(w[0]) + dy(w[1])) //
macro F11 [0.5*((-dx(phi1)+1)^2 - (-dx(phi2))^2), sqrt(2.)*(-dx(phi1)+1)*(-dx(phi2)), -0.5*((-dx(phi1)+1)^2 - (-dx(phi2))^2)] //
macro F12 [0.5*((-dx(phi1)+1)*(-dx(phi2)) - (-dy(phi2))*(-dy(phi2)+1)), (sqrt(2.)/2)*(-dx(phi2)^2 + (-dx(phi1)+1)*(-dy(phi2)+1)), -0.5*((-dx(phi1)+1)*(-dx(phi2)) - (-dy(phi2))*(-dy(phi2)+1))] //
macro F22 [0.5*((-dx(phi2))^2 - (-dy(phi2) + 1)^2), sqrt(2.)*(-dx(phi2))*(-dy(phi2) + 1), -0.5*((-dx(phi2))^2 - (-dy(phi2) + 1)^2)] //

// dynamic viscosity of water (@ 25 degrees Celcius) is 1.004
real nu=0.001;

//value of epsilon that will be used in the penalty term
real eps=0.000000001;

varf a11(w11,v)=int2d(Th)( 2*nu*(e(w11)'*e(v)) + (xi/eps)*(e(w11)'*e(v)) + 0.000001*(w11'*v) );
varf a12(w12,v)=int2d(Th)( 2*nu*(e(w12)'*e(v)) + (xi/eps)*(e(w12)'*e(v)) + 0.000001*(w12'*v) );
varf a22(w22,v)=int2d(Th)( 2*nu*(e(w22)'*e(v)) + (xi/eps)*(e(w22)'*e(v)) + 0.000001*(w22'*v) );

varf b11([p],w11)=int2d(Th)(p*div(w11));
varf b12([p],w12)=int2d(Th)(p*div(w12));
varf b22([p],w22)=int2d(Th)(p*div(w22));

varf m(p,q)=int2d(Th)(q);

varf h11(w11,v)=int2d(Th)(-0.00000123662*kappa*(F11'*e(v)));
varf h12(w12,v)=int2d(Th)(-0.00000123662*kappa*(F12'*e(v)));
varf h22(w22,v)=int2d(Th)(-0.00000123662*kappa*(F22'*e(v)));

matrix N11;
matrix N12;
matrix N22;
real[int] L11(Vh.ndof+Ph.ndof+1);
real[int] L12(Vh.ndof+Ph.ndof+1);
real[int] L22(Vh.ndof+Ph.ndof+1);
{
matrix A11=a11(Vh,Vh);
matrix B11=b11(Ph,Vh);
real[int] C=m(0,Ph);

N11=[[A11, B11, 0],
[B11', 0, C],
[0, C', 0]];

L11=0;

L11(0:Vh.ndof-1)=h11(0,Vh);}

{
matrix A12=a12(Vh,Vh);
matrix B12=b12(Ph,Vh);
real[int] C=m(0,Ph);

N12=[[A12, B12, 0],
[B12', 0, C],
[0, C', 0]];

L12=0;

L12(0:Vh.ndof-1)=h12(0,Vh);}

{
matrix A22=a22(Vh,Vh);
matrix B22=b22(Ph,Vh);
real[int] C=m(0,Ph);

N22=[[A22, B22, 0],
[B22', 0, C],
[0, C', 0]];

L22=0;

L22(0:Vh.ndof-1)=h22(0,Vh);}

Vh w11; Ph p; // Solving the system
{
set(N11,solver=UMFPACK);
real[int] sol=N11^-1*L11;
w11=[0,0];p=0;
w11[]=sol(0:Vh.ndof-1);
p[]=sol(Vh.ndof:Vh.ndof+Ph.ndof-1);

```

```

}

Vh w12;
{
set(N12,solver=UMFPACK);
real[int] sol=N11^-1*L12;
w12=[0,0];p=0;
w121[]=sol(0:Vh.ndof-1);
p[]=sol(Vh.ndof:Vh.ndof+Ph.ndof-1);
}

Vh w22;
{
set(N22,solver=UMFPACK);
real[int] sol=N11^-1*L22;
w22=[0,0];p=0;
w221[]=sol(0:Vh.ndof-1);
p[]=sol(Vh.ndof:Vh.ndof+Ph.ndof-1);
}

/////////////////////////////////////////////////////////////////
/////////////////////////////////////////////////////////////////

macro u11 [u11,u112] //
macro u12 [u121,u122] //
macro u22 [u221,u222] //
//macro v [v1,v2] //
//macro vec [1,1] //
//macro e(u) [dx(u[0]),(dy(u[0])+dx(u[1]))/sqrt(2.),dy(u[1])] //
//macro div(u) (dx(u[0]) + dy(u[1])) //
macro D11 [0.5, 0, -0.5] //
macro D12 [0, 0.5, 0] //
macro D22 [-0.5, 0, 0.5] //

varf aa11(u11,v)=int2d(Th)( 2*nu*(e(u11)*e(v)) + (xi/eps)*(e(u11)*e(v)) + 0.000001*(u11'*v) );
varf aa12(u12,v)=int2d(Th)( 2*nu*(e(u12)*e(v)) + (xi/eps)*(e(u12)*e(v)) + 0.000001*(u12'*v) );
varf aa22(u22,v)=int2d(Th)( 2*nu*(e(u22)*e(v)) + (xi/eps)*(e(u22)*e(v)) + 0.000001*(u22'*v) );
varf bb11([p],u11)=int2d(Th)(p*div(u11));
varf bb12([p],u12)=int2d(Th)(p*div(u12));
varf bb22([p],u22)=int2d(Th)(p*div(u22));
//varf m(p,q)=int2d(Th)(q);
varf hh11(u11,v)=-int2d(Th)((xi/eps)*(D11'*e(v)));
varf hh12(u12,v)=-int2d(Th)((xi/eps)*(D12'*e(v)));
varf hh22(u22,v)=-int2d(Th)((xi/eps)*(D22'*e(v)));

matrix NN11;
matrix NN12;
matrix NN22;
real[int] LL11(Vh.ndof+Ph.ndof+1);
real[int] LL12(Vh.ndof+Ph.ndof+1);
real[int] LL22(Vh.ndof+Ph.ndof+1);
{
matrix A11=aa11(Vh,Vh);
matrix B11=bb11(Ph,Vh);
real[int] C=m(0,Ph);

NN11=[[A11, B11, 0],
[B11', 0, C],
[0, C', 0]];

LL11=0;

LL11(0:Vh.ndof-1)=hh11(0,Vh);}

{
matrix A12=aa12(Vh,Vh);
matrix B12=bb12(Ph,Vh);
real[int] C=m(0,Ph);

NN12=[[A12, B12, 0],
[B12', 0, C],
[0, C', 0]];

LL12=0;

LL12(0:Vh.ndof-1)=hh12(0,Vh);}

{
matrix A22=aa22(Vh,Vh);
matrix B22=bb22(Ph,Vh);
real[int] C=m(0,Ph);

NN22=[[A22, B22, 0],
[B22', 0, C],
[0, C', 0]];

LL22=0;

```

```

LL22(0:Vh.ndof-1)=hh22(0,Vh);}

Vh u11; //Ph p; // Solving the system
{
set(NN11,solver=UMFPACK);
real[int] sol=NN11^-1*LL11;
u11=[0,0];p=0;
u11[]=sol(0:Vh.ndof-1);
p[]=sol(Vh.ndof:Vh.ndof+Ph.ndof-1);
}

Vh u12;
{
set(NN12,solver=UMFPACK);
real[int] sol=NN12^-1*LL12;
u12=[0,0];p=0;
u12[]=sol(0:Vh.ndof-1);
p[]=sol(Vh.ndof:Vh.ndof+Ph.ndof-1);
}

Vh u22;
{
set(NN22,solver=UMFPACK);
real[int] sol=NN22^-1*LL22;
u22=[0,0];p=0;
u22[]=sol(0:Vh.ndof-1);
p[]=sol(Vh.ndof:Vh.ndof+Ph.ndof-1);
}

Vh [ww11,ww112]=1.e7*w11; //use this for F11
Vh [ww121,ww122]=1.e4*w12; //use this for F12
Vh [ww221,ww222]=1.e7*w22; //use this for F22

Vh [uu111,uu112]=1.e1*u11;
Vh [uu121,uu122]=1.e1*u12;
Vh [uu221,uu222]=1.e1*u22;

//plot([ww11,ww112], wait=true, value=true, ps="xi_11.eps"); // Display
//plot([ww121,ww122], wait=true, value=true, ps="xi_12.eps"); // Display
//plot([ww221,ww222], wait=true, value=true, ps="xi_22.eps"); // Display

//plot([uu111,uu112], value=true, wait=true, ps="chi_11.eps"); // Display
//plot([uu121,uu122], value=true, wait=true, ps="chi_12.eps"); // Display
//plot([uu221,uu222], value=true, wait=true, ps="chi_22.eps"); // Display
//plot(u, wait=true, value=true); // Display

//real betab=0.5*( int2d(Th)( 2*nu*e(w11)*(e(u11)+D11) ) + int2d(Th)( 0.00000123662*kappa*F11*(e(u11)+D11) ) + int2d(Th)( kappa*F11[0] )
//+ int2d(Th)( 2*nu*e(w22)*(e(u22)+D22) ) + int2d(Th)( 0.00000123662*kappa*F22*(e(u22)+D22) ) - int2d(Th)( kappa*F22[0] )
//+ int2d(Th)( 2*nu*e(w22)*(e(u11)+D11) ) + int2d(Th)( 0.00000123662*kappa*F22*(e(u11)+D11) ) + int2d(Th)( kappa*F22[0] )
//+ int2d(Th)( 2*nu*e(w11)*(e(u22)+D22) ) + int2d(Th)( 0.00000123662*kappa*F11*(e(u22)+D22) ) - int2d(Th)( kappa*F11[0] ) );

real betas=int2d(Th)( 2*nu*e(w11)*(e(u11)+D11) ) + int2d(Th)( 0.00000123662*kappa*F11*(e(u11)+D11) ) + int2d(Th)( 0.00000123662*kappa*F11[0] )
+ int2d(Th)( 2*nu*e(w22)*(e(u22)+D22) ) + int2d(Th)( 0.00000123662*kappa*F22*(e(u22)+D22) ) - int2d(Th)( 0.00000123662*kappa*F22[0] )
+ 2*int2d(Th)( 2*nu*e(w12)*(e(u12)+D12) ) + int2d(Th)( 0.00000123662*kappa*F12*(e(u12)+D12) ) - int2d(Th)( 0.00000123662*kappa*F12[1] ) - 0.5*betab;

cout << " Beta_B is" << betab ;
cout << " Beta_S is" << betas ;

Vh psi,phi; // dy(u1),dx(u2) are polynomials of 1st

solve streamlines(psi,phi,solver=UMFPACK) =

int2d(Th)( dx(psi)*dx(phi) + dy(psi)*dy(phi) + 0.000001*phi*psi)
+ int2d(Th)( (dx(w112)-dy(w111))*phi );

plot(psi,nbiso=20,ps="s_xi_22.eps");

```

□

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